Sparse Time-Frequency Representation for Signals with Fast Varying Instantaneous Frequency

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Abstract: Time-frequency distributions have been used for providing high resolution representation in a large number of signal processing applications. However, high resolution and accurate instantaneous frequency (IF) estimation usually depends on the employed distribution and complexity of signal phase function. To ensure an efficient IF tracking for various types of signals, a class of complex-time distributions has been developed. These distributions facilitate analysis in cases when standard distributions cannot provide satisfactory results (e.g., for highly non-stationary signal phase). In that sense, an ambiguity based form of the forth order complex-time distribution is considered, in a new compressive sensing (CS) context. CS is an intensively growing approach in signal processing that allows efficient analysis and reconstruction of randomly under-sampled signals. In this paper, randomly chosen ambiguity domain coefficients serve as CS measurements. By exploiting sparsity in the time-frequency plane, it is possible to obtain highly concentrated IF using just small number of randomly chosen coefficients from the ambiguity domain. Moreover, in noisy signal case, this CS approach can be efficiently combined with the L-statistics producing robust time-frequency representations. Noisy coefficients are firstly removed using the L-statistics and then reconstructed by using the CS algorithms. The theoretical considerations are illustrated using experimental results.

1. Introduction

Highly-localized signal IF in the time-frequency (TF) plane has attracted attention of researchers and has been widely studied in the literature [1]-[6]. In order to deal with different types of signals, numerous TF distributions have been developed [7]. The quadratic distributions (Wigner-Ville distribution – WD and distributions from the Cohen class) can perfectly localize linear frequency modulated signals. Therefore, in order to reduce or eliminate the cross-terms in the WD, distributions from the Cohen class are employed [5]-[8]. These are based on low-pass filtering of the ambiguity function, which is two-dimensional Fourier transform of the WD. Having auto-terms around the origin and cross-terms dislocated from the origin in the ambiguity plane, resulting TF distribution has a reduced number of cross-terms, or is cross-terms free. However, in general there is a trade-off between localization in the TF plane and cross-terms reduction, which is known as the TF uncertainty principle [4].

Signals such as radar signals or vibrating tones of musical instruments, are characterized by the fast variations of the IF [9], [10]. These variations contain information about the related physical phenomena or the radar target, which can be efficiently analysed and extracted using the TF representations [7]. An interesting application of the TF representations assumes characterization of the micro-Doppler effect that appears in radars [11]-[15]. It is produced by the fast moving parts of the target and can be used for target recognition [15]. When dealing with human walking, fast variations caused by micro movements can be hardly detected using standard quadratic TF representations (such as the WD, the S-method or the Cohen class distributions). Furthermore, modifications of the S-method (adaptive and multi-window form), improve the concentration in the TF plane, but still keep the limitations of the quadratic nature. Particularly, there will be a significant influence of higher order phase derivatives in the case of considered fast-varying signals (e.g., derivatives of sine/cosine signal modulations caused by rotating target parts). Therefore, the complex-time distributions [16]-[25] appear as an optimal choice for characterization of fast varying data, providing high concentration along IF, and consequently, the exact IF tracking.

The complex-time distributions of different orders are applicable to various types of signals, depending on the phase function. The focus of this paper is on the fourth order complex-time distribution, based on the ambiguity domain realization. Most of the real signals exhibit sparsity property in a certain domain (time, frequency, time-frequency, etc.) [1], [2]. Sparsity means that, in certain domain, the signal could be represented with a small number of non-zero coefficients. Sparse signals are the subject of CS application, which is a new and intensively studied topic in signal processing [26]-[46]. CS uses signals sampled at the rates below Nyquist, and provides successful signal reconstruction using small set of randomly selected samples [27]. Random sampling is a necessary condition that should be satisfied, in order to reconstruct signal from incomplete set of samples. Most of the real-world signals are sparse in one domain, while they are dense in the other [28]. Here, we deal with signals that are sparse in the joint TF domain, i.e., signals whose IF occupies only small part of the TF plane [1], [2], [34], [38]. By exploiting sparsity in the TF domain, the CS approach is applied to the signal in order to provide better IF localization with reduced number of samples.

It was shown that the ambiguity function can be combined with CS to provide sparse TF representation, especially for linear IF [34], [38]. The observations are used from the ambiguity domain, usually from a priori defined area. The shape and size of the area is defined by the mask that should be large enough to collect auto-terms and small enough to avoid the cross-terms. Here we deal with quite a complex signal structure, and thus we explore the use of ambiguity based complex-time distribution in order to provide sparse and cross-terms free representation and to facilitate samples acquisition. In the cases of noisy signals, the robust form is provided by using the L-statistics.

The paper is organized as follows. The theoretical background on commonly used TF distributions is given in the Section II. Distributions with the complex-time argument are described in this section, as well. Section III presents the CS in the TF domain. A robust approach for the IF estimation by using complex-time distributions and CS is described in Section IV. Section V contains several examples which justify presented theory. Conclusions are given in Section VI.

2. A review of time-frequency distributions

Real-world signals differ in their nature: stationarity, sparsity, number of components present in the signal, etc. Therefore, different tools and methods have been used for signal processing and analysis. The target signals in this paper are non-stationary signals with fast oscillations of spectral content, which entail TF based approaches [7]. The commonly used quadratic TF distribution is the WD defined as:

$$WD(t,\omega)=\int_{-\infty}^{\infty} x(t+\frac{\tau}{2})x^*(t-\frac{\tau}{2})e^{-j\omega\tau}d\tau. \qquad (1)$$

In the case of multicomponent signals the WD produces the cross-terms, while the spread factor will contain all odd phase derivatives in the cases of nonlinear IF changes. The Cohen class distributions, the S-method, and the distributions with "complex-time argument", are introduced with the aim to overcome drawbacks of the WD. We will focus on the ambiguity domain based distributions, i.e., the Cohen class and class of "complex-time" distributions. The ambiguity function (AF) is defined as a two-dimensional Fourier transform of the WD:

$$A(\theta,\tau)=FT_{2D}\{WD(t,\omega)\}=\int_{-\infty}^{\infty} x(t+\frac{\tau}{2})x^*(t-\frac{\tau}{2})e^{-j\theta t}dt, \qquad (2)$$

where FT_{2D} denotes the 2D Fourier transform. Having in mind that signal terms are located around the origin in the ambiguity plane and that the cross-terms are dislocated, they can be easily reduced or completely removed by applying the low-pass kernel function. It leads to the distributions from the Cohen class defined as follows:

$$CD(t,\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(\theta,\tau)A(\theta,\tau)e^{-j\theta t-j\omega\tau}d\tau d\theta, \qquad (3)$$

where $K(\theta,\tau)$ denotes a 2D kernel function.

However, in the cases of the signals with fast varying IF, the Cohen class distributions cannot provide satisfactory results in tracking the IF changes. Therefore, higher order distributions based on the complex-time argument should be used to provide high concentration in the TF plane, and consequently, better IF estimation for signals with fast varying IF.

The 4th order complex-time distribution can be defined as:

$$CTD_4(t,\omega)=\int_{-\infty}^{\infty}\prod_{i=1}^{2}x(t+\frac{\tau}{4(a_i+jb_i)})^{(a_i+jb_i)}x(t-\frac{\tau}{4(a_i+jb_i)})^{-(a_i+jb_i)}e^{-j\omega\tau}d\tau\,, \tag{4}$$

where a_i and b_i denote points on the unit circle. Let us consider the case where (a_1, b_1, a_2, b_2)=(1,0,0,1). The corresponding 4th order complex-time distribution is given in the form:

$$CTD_4(t,\omega)=\int_{-\infty}^{\infty}x(t+\frac{\tau}{4})x^{-1}(t-\frac{\tau}{4})x^{j}(t-j\frac{\tau}{4})x^{-j}(t+j\frac{\tau}{4})e^{-j\omega\tau}d\tau, \tag{5}$$

and the corresponding spread factor is:

$$S(t,\tau)=\phi^{(5)}(t)\frac{\tau^5}{4^4 5!}+\phi^{(9)}(t)\frac{\tau^9}{4^8 9!}+... \tag{6}$$

Starting from the 4th order moment function, it is possible to separate the real-time and complex-time ambiguity functions, as follows [6]:

$$\begin{aligned}A_{rt}(\theta,\tau)&=\int_{-\infty}^{\infty}x(t+\frac{\tau}{4})x^{*}(t-\frac{\tau}{4})e^{-j\theta t}dt,\\ A_{ct}(\theta,\tau)&=\int_{-\infty}^{\infty}x^{-j}(t+j\frac{\tau}{4})x^{j}(t-j\frac{\tau}{4})e^{-j\theta t}dt.\end{aligned} \tag{7}$$

Both of the ambiguity functions can be filtered using the kernel function:

$$\begin{aligned}A_{rt}^{K}(\theta,\tau)&=K(\theta,\tau)A_{rt}(\theta,\tau),\\ A_{ct}^{K}(\theta,\tau)&=K(\theta,\tau)A_{ct}(\theta,\tau).\end{aligned} \tag{8}$$

The resulting ambiguity function is obtained as [6]:

$$A(\theta,\tau)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}W(\varepsilon)e^{-j\varepsilon\tau_1}e^{j\varepsilon(\tau-\tau_1)}A_{rt}^{K}(\theta_1,\tau_1)A_{ct}^{K}(\theta-\theta_1,\tau-\tau_1)d\tau_1 d\theta_1 d\varepsilon, \tag{9}$$

where $W(\varepsilon)$ is a window function. Now, the complex-time distribution can be calculated as:

$$CTD(t,\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}A(\theta,\tau)e^{j\theta t-j\omega\tau}d\tau d\theta\,. \tag{10}$$

For signals with higher order phase nonstationarities, the distribution order can be increased to diminish the influence of higher order phase derivatives and to improve TF concentration.

In the sequel, we propose an approach to combine the complex-time distribution and the CS approach. This combination can be used to provide a sparse TF representation and an efficient IF estimation.

3. Time-Frequency representations and Compressive Sensing

a. *Compressive Sensing concept*

Compressive Sensing (CS) approach [26] has been widely studied in the recent years. It provides successful signal reconstruction using an incomplete set of signal samples, i.e. deals with signals sampled at the rate lower than Nyquist. Signal can be intentionally under-sampled. In the noisy signal cases, corrupted samples can be considered as missing ones, if we are able to locate these samples in the signal. In order to apply CS procedure, certain conditions should be satisfied, such as sparsity and incoherence between measurement and transform matrix [35], [36]. Sparsity refers to the property that the signal can be represented by a small number of non-zero coefficients in certain transform basis. Incoherent sensing provides successful signal reconstruction using a small set of signal samples. Many signals in real application satisfy these two conditions, allowing wide applications of CS.

The signal reconstruction from a small number of available samples is done using complex optimization algorithms. Assume that we have a time domain signal x of length N. Let $\mathcal{F}^{-1}$ be sparsifying basis for the signal x. Therefore, signal can be represented in the form of the basis matrix as follows:

$$x = \mathcal{F}_{N\times N}^{-1} S \,. \tag{11}$$

If $\mathcal{F}_{N\times N}^{-1}$ denotes $N\times N$ inverse Fourier transform matrix, then S denotes the vector of Fourier transform coefficients. In the matrix form, previous relation becomes:

$$\begin{bmatrix} x(0) \\ x(1) \\ \dots \\ x(N-1) \end{bmatrix} = \frac{1}{N} \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & e^{j\frac{2\pi}{N}} & \dots & e^{j\frac{2(N-1)\pi}{N}} \\ \dots & \dots & \dots & \dots \\ 1 & e^{j\frac{2(N-1)\pi}{N}} & \dots & e^{j\frac{2(N-1)(N-1)\pi}{N}} \end{bmatrix} \begin{bmatrix} S(0) \\ S(1) \\ \dots \\ S(N-1) \end{bmatrix}, \tag{12}$$

where each element of the matrix $\mathcal{F}_{N\times N}^{-1}$ is an exponential term $e^{j(2\pi kn/N)}$, $k = 0,...,N-1;\ n = 0,...,N-1$. Assume now that only M randomly distributed samples of the signal x are known, while the rest (N-M) samples are considered as missing. This means that the Fourier transform matrix is not a full $N\times N$ matrix, but rather a randomly subsampled matrix that contains M out of N randomly selected rows of the original matrix. Random selection can be modelled as a matrix multiplication of the original matrix $\mathcal{F}_{N\times N}^{-1}$ with the incoherent measurement matrix $\Phi_{M\times N}$. Matrix formed in such a way is called random partial Fourier matrix, and can be defined as:

$$\mathcal{F}_P^{-1} = \Phi_{M\times N}\mathcal{F}_{N\times N}^{-1} = \frac{1}{N}\begin{bmatrix} 1 & e^{jn_1\frac{2\pi}{N}} & \dots & e^{jn_1\frac{2\pi}{N}(N-1)} \\ 1 & e^{jn_2\frac{2\pi}{N}} & \dots & e^{jn_2\frac{2\pi}{N}(N\text{-}1)} \\ \dots & \dots & \dots & \dots \\ 1 & e^{jn_M\frac{2\pi}{N}} & \dots & e^{jn_M\frac{2\pi}{N}(N\text{-}1)} \end{bmatrix}, \tag{13}$$

where (n_1, n_2,…, n_M) correspond to the positions of the available samples. Randomly chosen M rows of the matrix $\mathcal{F}_{N\times N}^{-1}$ result in the measurement vector v:

$$v = \mathcal{F}_P^{-1}S\,. \tag{14}$$

The system of equations (14) is undetermined. In order to find the sparsest solution, among large number of possible solutions, different optimization algorithms are used: greedy algorithms (MP, OMP, StOMP, CoSaMP, etc.) [39]-[42], convex relaxation algorithms and the least absolute shrinkage and selection operator (LASSO) [43], non-iterative and iterative solutions [29], [45], etc. Commonly used optimization algorithms are based on ℓ_1-norm minimization [39]:

$$\min_s \|S\|_{\ell_1} \text{ subject to } \quad v = \mathcal{F}_P^{-1}S\,. \tag{15}$$

b. Extended problem formulation – 2D partial Fourier transform matrix

The CS approach can be applied to the TF domain in order to provide sparse representation and better localization of the signal in the TF plane [34], [38]. When TF domain is considered as a domain of sparsity, then the ambiguity domain can be used as a domain of observations. Hence, the measurements are randomly selected from the ambiguity domain.

The relationship between the ambiguity function and complex-time distribution can be written as:

$$A(\theta,\tau) = \mathcal{F}^{2D}\cdot CTD(t,\omega)\,, \tag{16}$$

where CTD denotes complex-time distribution, and $\mathcal{F}^{2D}$ is 2D $N^2\times N^2$ Fourier transform matrix. Therefore, we assume that the dense counterpart of the ambiguity domain is the 4th order complex-time distribution CTD. Matrix $\mathcal{F}^{2D}$ is produced as the Kronecker product of the identity matrix $\boldsymbol{I}$ and 1D Fourier transform matrix as follows:

$$\mathcal{F}_{N^2\times N^2}^{2D} = I_{N\times N}\otimes\mathcal{F}_{N\times N}^{1D}\,, \tag{17}$$

where $\otimes$ denotes the Kronecker product. In the matrix form, we can write:

$$\begin{bmatrix} \mathcal{F}^{1D}_{N\times N} & 0 & \dots & 0 \\ 0 & \mathcal{F}^{1D}_{N\times N} & \dots & 0 \\ \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & \mathcal{F}^{1D}_{N\times N} \end{bmatrix} = \begin{bmatrix} 1_{N\times N} & 0 & \dots & 0 \\ 0 & 1_{N\times N} & \dots & 0 \\ \dots & \dots & \dots & 0 \\ 0 & 0 & 0 & 1_{N\times N} \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 & \dots & 1 \\ 1 & e^{-j\frac{2\pi}{N}} & \dots & e^{-j\frac{2(N-1)\pi}{N}} \\ \dots & \dots & \dots & \dots \\ 1 & e^{-j\frac{2(N-1)\pi}{N}} & \dots & e^{-j\frac{2(N-1)^2\pi}{N}} \end{bmatrix}. \tag{18}$$

Now we can define the CS problem in the ambiguity domain. For $N\times N$ TF representation, at most $K\times N$ non-zero components should exist, where K is the number of signal components [34], [38]. If we consider only the coefficients around the origin in the ambiguity plane, and use these coefficients as measurements for the CS procedure, the obtained ambiguity function is a measurement ambiguity function. It can be described as follows:

$$\mathbf{A}^M = \begin{bmatrix} \mathbf{0}_{L\times L} & \mathbf{0}_{L\times J} & \mathbf{0}_{L\times L} \\ \mathbf{0}_{J\times L} & \mathbf{\Phi}_{J\times J} & \mathbf{0}_{J\times L} \\ \mathbf{0}_{L\times L} & \mathbf{0}_{L\times J} & \mathbf{0}_{L\times L} \end{bmatrix} \mathbf{A}. \tag{19}$$

The matrix Φ is of size $J\times J$ (where $J+2L=N$), and contains randomly positioned $p\%$ of values "1" and (100-$p\%$) of values "0". In the sequel, it will be referred to as the mask in the ambiguity domain, while $\mathbf{A}^M$ (vector of size $J^2\times 1$) will be referred as the masked ambiguity function. Note that the ambiguity function is firstly rearranged into vector $\mathbf{A}$ of size $N^2\times 1$.

The sparse TF distribution can be obtained by minimizing the following function:

$$\min_{\sigma}\left(F(\sigma)+G(\sigma)\right), \tag{20}$$

where F performs soft thresholding according to the relation: $F(\sigma)=\max(0,1-\frac{\lambda}{|\sigma|})\sigma$, λ is a regularization parameter, and $G(\sigma)$ is defined as:

$$G(\sigma)=\mathcal{F}_P^{'}(\mathcal{F}_P\,\sigma - A^M). \tag{21}$$

$\mathcal{F}_P$ denotes partial 2D Fourier transform matrix, $\mathcal{F}_P^{'}$ is its transpose and σ is a row vector which is, after the optimization problem is solved, rearranged into the matrix and forms a sparse TF representation. Matrix $\mathcal{F}_P$ is a row-subsampled matrix as only those rows which correspond to the positions of measurements from the mask are kept, while the other rows are discarded.

4. Robust approach to the IF estimation

In the cases where the ambiguity function is corrupted by impulsive noise, the randomly selected coefficients used as measurements in the CS procedure, will be noisy as well. The initial transform domain vector, used in CS optimization problem, should be noise free. Therefore, the robust statistics can be applied to the coefficients of the ambiguity function in order to remove noisy peaks. This is achieved by using the L-statistics that performs well in the presence of impulsive and mixed noise [34], [38].

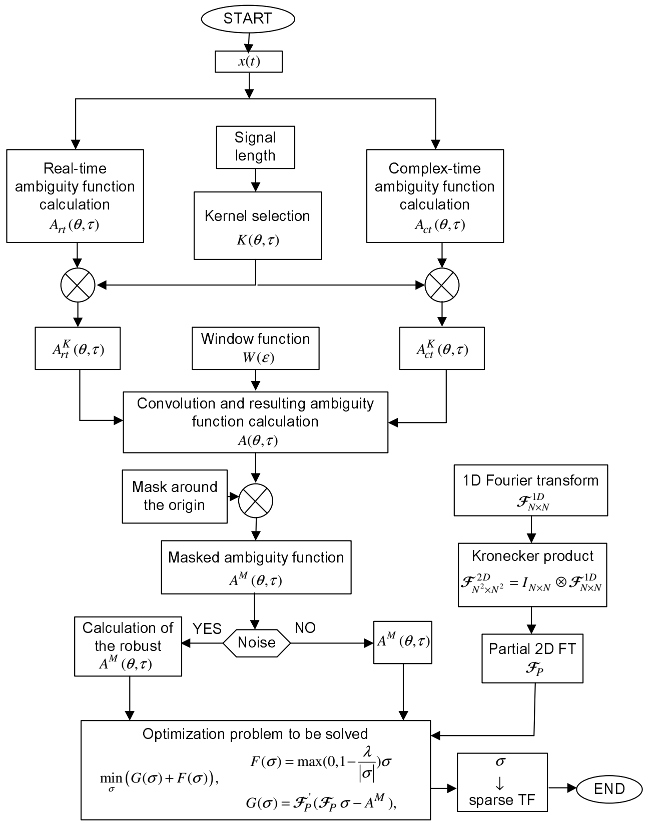

Fig. 1. *Flowchart of the proposed algorithm*

According to the L-estimation approach, the noisy measurement vector is sorted, and certain percent of the smallest and the largest value coefficients is discarded. Therefore, the L-estimation based minimization problem can be defined as follows:

$$\arg\min_{S} \|S\|_{\ell_1} \quad \text{subject to} \qquad v = \mathcal{F}_P^{-1} S \,. \tag{22}$$

A robust initial transform is obtained as:

$$s_0 = \sum_{i=P}^{Q} A_{SORT},$$
$$A_{SORT}(\tau,\theta) = sort\{A(\tau,\theta)e^{-j2\pi\tau k/M}e^{-j2\pi\theta l/M}\} \quad (23)$$

where P and Q denote the number of discarded coefficients, smallest and largest respectively. The proposed algorithm for obtaining sparse TF representation is represented by the flowchart in Figure 1.

5. Experimental results

Example 1: Simulated radar signal with fast varying IF changes

Consider the multicomponent signal with the nonlinear phase function in the form:

$$x(t) = e^{j(2\cos(\pi t)+\cos(4\pi t)+4.5\pi t)/2} + \\ + e^{j(\cos(\pi t)+\cos(3\pi t)+\cos(4\pi t)-8\pi t)/2}. \quad (24)$$

In order to find a suitable TF representation for the observed signal, several TF representations are considered. The TF representations are of 90×90 size and they are displayed in Figure 2a-2c. Firstly, the WD distribution is calculated.

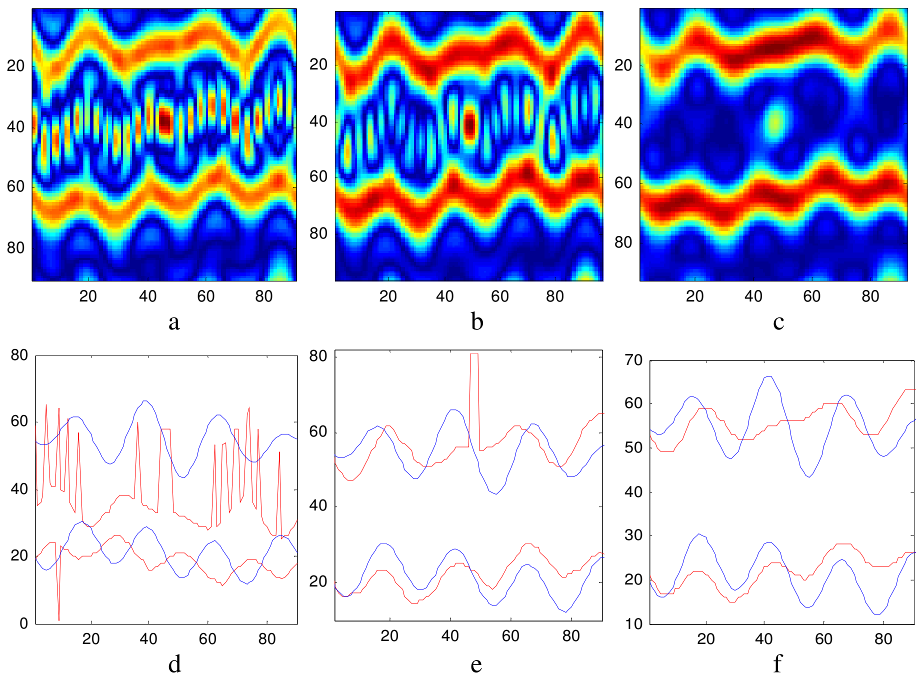

Fig. 2. *a. The WD of the signal (24); b.-c. The Cohen class distributions, obtained by using Gaussian kernel with: b. δ =80 and c. δ=20 d. IF estimated from the WD; e. and f. IFs estimated from the Cohen class distributions with Gaussian kernel: e. δ=80 and f. δ=20. Blue line denotes the correct IF and red line corresponds to the estimated IF*

We may observe that the WD cannot follow fast IF variations (see Figure 2a and 2d), and, in addition, it introduces the cross-terms. Therefore, different Cohen class distributions are tested to avoid cross-terms. As a kernel function, the Gaussian low-pass filtering function is used for different values of parameter δ (distributions obtained with two different values: δ=80 and δ=20 are shown in Figure 2b and 2c). Mean square errors (MSE) and relative mean square errors (RMSE) for different TF distributions and different number of CS measurements are given in Table 1. The distributions from the Cohen class enable controlling of the cross-terms amount, but again, fail to provide accurate IF tracking of the non-stationary signals, as it is shown in Figure 2e and 2f.

Table 1 MSEs of the IF estimation

Distribution	MSE		Relative MSE -RMSE (%)	
	Component 1	Component 2	Component 1	Component 2
Wigner distribution	3.3192×10^3	79.7761	67.81	5.79
Distribution from the Cohen class based on Gaussian kernel $e^{-(\tau^2+\theta^2)/\delta^2}$ with δ=120	1.1563×10^3	3.1624×10^3	28.3	77.43
Distribution from the Cohen class based on Gaussian kernel $e^{-(\tau^2+\theta^2)/\delta^2}$ with δ=80	1.3880×10^3	1.5905e$\times10^3$	71.59	78.04
Distribution from the Cohen class based on Gaussian kernel $e^{-(\tau^2+\theta^2)/\delta^2}$ with δ=20	1.6176e$\times10^3$	1.5584e$\times10^3$	61.02	54.82
Fourth order complex-time distribution:				
7×7 mask and all samples within the mask used	failed	failed	/	/
10×10 mask and all samples within the mask used	failed	Failed	/	/
15×15 mask and 70% samples used	81.3099	57.1962	11.08	0.02
20×20 mask and 60% samples used	8.9314	20.5861	0.25	3.43
25×25 mask				
40% samples from the mask used	17.7834	32.7570	1.23	3.48
50% samples from the mask used	10.7177	9.2834	1.09	0.36
60% samples from the mask used	7.3854	8.1211	0.30	0.31

The IFs estimated from the observed distributions, are shown in Figure 2 with red line, while the correct IF is shown with blue line. CS based TF distributions, calculated starting from the WD and the Cohen class distributions, are shown in Figure 3. For the CS based TF distribution, the TF mask is formed by using central region of 25×25 size in the ambiguity domain. This region contains 7.7% of the total number of samples. The amount of 50% of measurements is chosen randomly from the mask (3.8% of the total number of samples). Note that the resulting distributions are not sparse and do not follow accurately the IF changes. Therefore, the complex-time CS distribution is calculated. The standard form of the 4^{th} order complex-time distribution is shown in Figure 4a. The ambiguity domain filtering with Gaussian kernel is used for the calculation of the complex-time distribution. The mask size is 25×25. The resulting sparse TF distribution (Figure 4b) is provided by using 60% of the samples from the mask, which is around 5% of the total number of samples. Very similar results are obtained even for a lower number of measurements, e.g., 40%. The IFs estimated from the sparse distributions, shown in Figure 4a and 4b, are very close to the true IF of the signal (see Figure 4c and 4d). Let us consider the influence of the mask size ($J\times J$) on the number of samples required to provide sparse TF representation. Figure 5 shows that the masks of size 7×7 and 10×10 are too small to provide accurate IF representation, even if all samples in the masks are used as CS measurements.

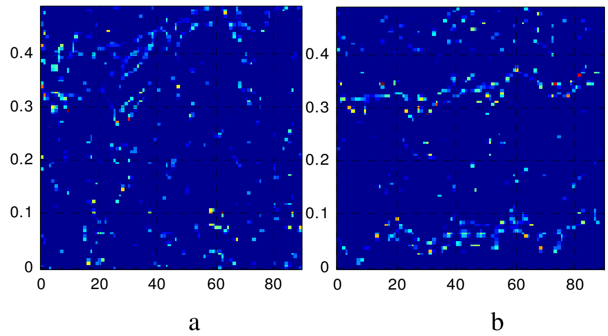

Fig. 3. *Sparse TF distributions obtained from: a. the WD, b. the Cohen class distribution based on Gaussian kernel*

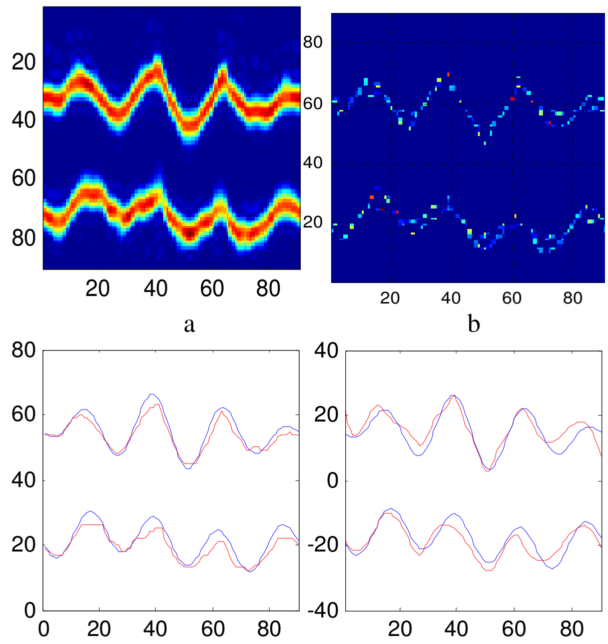

Fig. 4. *a. The 4th order complex-time distribution;*
b. Sparse time-frequency representation obtained by using 60% of randomly chosen measurements from 25×25 mask;
c. Red line: the IF estimated from the distribution in Fig. 4a, blue line-the true IF; d. The IF estimated from the distribution in Fig. 4b: blue line is the true IF, red line – the estimated IF

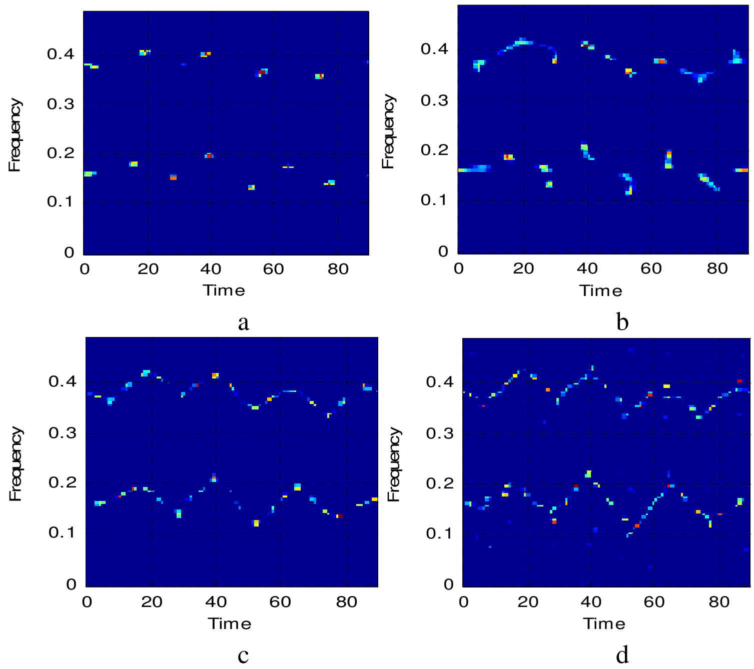

Fig. 5. *Sparse time-frequency distributions obtained by using different mask sizes and different percentages of used samples:*

a. 7×7 mask with 100% of samples;

b. 10×10 mask with 100% of samples;

c. 15×15 mask with 70% of samples;

d. 20×20 mask with 60% of samples

Larger masks (15×15 and 20×20) can provide an accurate IF estimation, but require a larger number of samples to be used as CS measurements (i.e. 70% and 60% of the measurements from the mask are used, respectively) in order to obtain an accurate IF estimation.

Example 2: Noisy signal measurements

Let us now consider the simulated radar signal. Monocomponent signal is frequency modulated periodically and has fast IF changes.

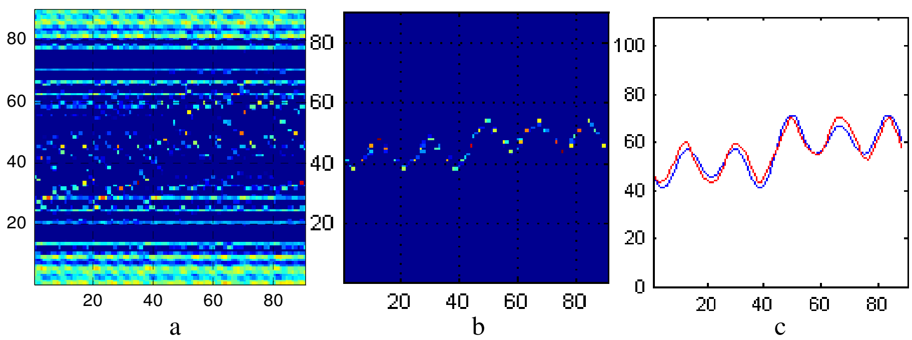

Fig. 6. *a. TF distribution obtained by using noisy ambiguity measurements;*

b. Sparse TF distribution obtained by using robust ambiguity function;

c. blue - true and red – estimated IF of the signal, obtained by using sparse TF distribution from the Fig. 6b

Non-uniform rotation of the reflecting point in radar systems could be described using this form of the signal [9]. The signal lasts for 96 seconds. It is sine modulated and sampled at frequency 48Hz: $x(t)=e^{j(4cos(\pi t)+(2/3)cos(3\pi t)+(2/3)cos(5\pi t))}$. The assumption is that the ambiguity domain measurements are corrupted by impulse noise. The TF distribution, obtained by using 50% of noisy ambiguity measurements from the mask of size 25×25, is shown in Figure 6a.

Table 2 MSE and RMSE of the IF estimation, for different SNR values

SNR	MSE	RMSE (%)
-62.0748	18.418	1.22
-50.0336	8.3441	0.23
-39.5760	9.5985	0.21
-30.0336	3.0854	0.18
-23.1703	4.9059	0.15

The ambiguity function of the signal is corrupted by the impulse noise, which disables accurate IF estimation of the observed signal. The CS based TF distribution (Figure 6a), obtained by using noisy ambiguity measurements, is not sparse and does not provide accurate IF estimation. Therefore, the L-estimation approach is applied to the ambiguity function, prior to the CS based TF distribution calculation. Namely, 0.5% of the highest ambiguity coefficients are removed and the robust form of the ambiguity function is obtained. The noise is successfully removed and a sparse TF is provided (Figure 6b). The estimated IF is very close to the true IF of the signal (Figure 6c). The performance of the proposed algorithm is tested for different values of SNR. The measurements are corrupted by impulse noise, and consequently 0.5% of noisy coefficients are removed in the ambiguity domain (as a consequence of L-estimation). The estimation errors are given in Table 2 (the MSEs and the RMSEs). In all considered cases, the successful IF estimation is achieved.

Example 3: Real radar signal

Finally, as an example of signal recorded in real noisy environment, let us observe a portion of radar data corresponding to the moving human body. The micro movements, appearing during the human walking, are hard to detect using the standard TF representations. In that sense, the high-resolution solutions provided by the complex-time distributions allow detection of the smallest variations. The radar producing the signal operates at 2.4 GHz carrier frequency, and transmitted power level is 5 dBm. The sampling frequency is 1 KHz. The ambiguity function of the observed signal is shown in Figure 7a. Figure 7b shows spectrogram while the complex-time distribution of the signal is shown in Figure 7c.

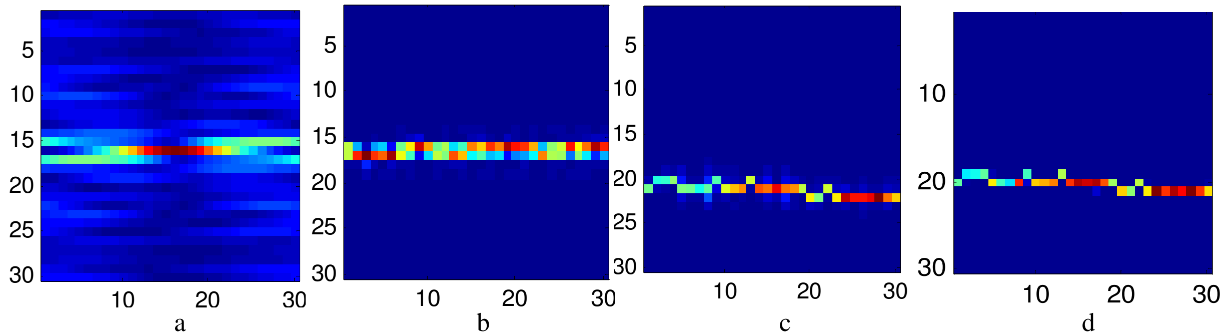

Fig. 7. *a. The ambiguity function of the real radar signal;*
b. The spectrogram;
c. The complex-time distribution of the signal;
d. The sparse complex-time distribution

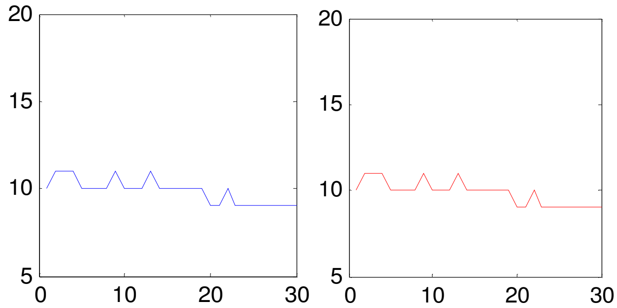

Fig. 8. *Estimated IFs: a. from the original complex-time distribution;*
b. from the sparse complex-time distribution

Now, assume that we are faced with a reduced set of available observations (11% available samples of the total number of samples in TF plane), and we need to employ the proposed CS based approach. The resulted sparse TF distribution is shown in Figure 7d, while the estimated IFs from the original and sparse complex-time distributions are shown in Figure 8a and 8b. As it can be seen from Figure 8, the IF estimated from the sparse TF corresponds to the IF estimated from the original complex-time distribution.

6. Conclusion

This paper deals with the IF estimation using CS based sparse TF representation. The non-stationary signals with fast-varying phase functions are observed. Having in mind that the standard distributions fail to accurately estimate the IF of observed signals, we employ the complex-time distributions. The CS observations are taken randomly from the masked ambiguity function calculated on a basis of the complex-time distribution. It has been shown that the sparse TF representation can be reconstructed from approximately 4% of the ambiguity domain measurements. Furthermore, the accuracy of IF estimation based on the resulting sparse TF distribution is proved by using the MSE.

7. References

[1] Borgnat, P., Flandrin, P.: 'Time-frequency localization from sparsity constraints', Proceedings of IEEE International Conference on Acoustics, Speech and Signal Proceesing ICASSP-08, Las Vegas (NV), April 2008, pp. 3785–3788

[2] Flandrin, P., Borgnat, P.: 'Sparse time-frequency distributions of chirps from a compressed sensing perspective', Proceedings of 8th IMA International Conference on Mathematics in Signal Processing, Cirencester (UK), December 2008, pp. 84–87

[3] Orović, I., Orlandić, M., Stanković, S., Uskoković, Z.: 'A Virtual Instrument for Time-Frequency Analysis of Signals with Highly Non-Stationary Instantaneous Frequency', IEEE Transactions on Instrumentation and Measurements, 2011, **60**, (3), pp. 791 – 803

[4] Flandrin, P., Borgnat, P.: 'Time-Frequency Energy Distributions Meet Compressed Sensing', IEEE Transactions on Signal Processing, 2010, **58**, (6), pp. 2974-2982

[5] Choi, H., Williams, W.: 'Improved time-frequency representation of multicomponent signals using exponential kernels', IEEE Transactions On Acoustics and Speech, 1989, **37**, (6), pp. 862-871

[6] Orović, I., Stanković, S.: 'A Class of Highly Concentrated Time-Frequency Distributions Based on the Ambiguity Domain Representation and Complex-Lag Moment', EURASIP Journal on Advances in Signal Processing, 2009, Article ID 935314, 9 pages

[7] Stanković, LJ., Daković, M., Thayaparan, T.: 'Time-Frequency Signal Analysis with Applications' (Artech House, Boston, London, 2013)

[8] Figueiredo, A. C. A., Nave, M. F. F.: 'Time–frequency analysis of nonstationary fusion plasma signals: A comparison between the Choi–Williams distribution and wavelets', Review of Scientific Instruments, 2004, **75**, (10), pp. 4268-4270

[9] Cornu, C., Stanković, S., Ioana, C., Quinquis, A., Stanković, LJ.: 'Generalized Representation of Phase Derivatives for Regular Signals', IEEE Transactions on Signal Processing, 2007, **55**, (10), pp. 4831-4838

[10] Wang, R., Jiang, Y.: 'ISAR Imaging of Ship Based on the Modified 4th Order Time-Frequency Distributions with Complex-Lag Argument', 2012 2nd International Conference on Remote Sensing, Environment and Transportation Engineering (RSETE), June 2012, pp.1-4

[11] Chen, V.C.: 'The Micro-Doppler Effect in Radar' (Artech House, Boston, London, 2011)

[12] Thayaparan, T., Stanković, LJ., Djurović, I.: 'Micro-Doppler Based Target Detection and Feature Extraction in Indoor and Outdoor Environments', Journal of the Franklin Institute, 2008, **345**, (6), pp 700-722

[13] Thayaparan, T., Abrol, S., Riseborough, E., Stankovic, LJ., Lamothe, D., Duff, G.: 'Analysis of Radar Micro-Doppler Signatures From Experimental Helicopter and Human Data', IET Proceedings Radar Sonar and Navigation, 2007, **1**, (4), pp. 288-299

[14] Chen, V. C., Tahmoush, D., Miceli, W.J.: 'Radar Micro-Doppler Signature: Processing and Applications' (IET digital library, 2014)

[15] Balleri, A., Chetty, K., Woodbridge, K.: 'Classification of personnel targets by acoustic micro-doppler signatures', IET Radar, Sonar and Navigation, 2011, **5**, (9), pp. 943-951

[16] Stanković, S., Stanković, LJ.: 'Introducing time-frequency distribution with a "complex-time" argument', Electronics Letters, 1996, **32**, (14), pp.1265-1267

[17] Stanković, S., Zarić, N., Orović, I., Ioana, C.: 'General form of time-frequency distribution with complex-lag argument', Electronics Letters, 2008, **44**, (11), pp. 699-701

[18] Wang, Y., Jiang, Y.: 'ISAR Imaging of Maneuvering Target Based on the L-Class of Fourth-Order Complex-Lag PWVD', IEEE Transactions on Geoscience and Remote Sensing, 2010, **48**, (3), pp.1518-1527

[19] Wang, Y., Jiang, Y.: 'Fourth-Order Complex-Lag PWVD for Multicomponent Signals with Application in ISAR Imaging of Maneuvering Targets', Circuits, Systems and Signal Processing, 2010, **29**, (3), pp. 449-457

[20] Rajshekhar, G., Rastogi P.: 'Application of complex-lag distributions for estimation of arbitrary order phase derivatives in digital holographic interferometry', Optics Letters, 2011, **36**, (19), pp. 3738-3740

[21] Stanković, S., Orović, I., Ioana, C.: 'Effects of Cauchy Integral Formula Discretization on the Precision of IF Estimation: Unified Approach to Complex-lag Distribution and its L-Form', IEEE Signal Processing Letters, 2009, **16**, (4), pp. 307-310

[22] Stanković, S., Orović, I.: 'Time-Frequency Rate Distributions with Complex-lag Argument', Electronics Letters, 2010, **46**, (13), pp. 950 - 952

[23] Morelande, M., Senadji, B., Boashash, B.: 'Complex-lag polynomial Wigner-Ville distribution', TENCON '97, IEEE Region 10 Annual Conference on Speech and Image Technologies for Computing and Telecommunications, Brisbane, Australia, December 1997, pp.43-46

[24] Bernard, C., Ioana, C.: 'Generalized Complex time-distribution using modified analytical continuation', Proceedings of the 21st European Signal Processing Conference (EUSIPCO), Marrakech, Morocco, September 2013, pp.1-5

[25] Zarić, N., Orović, I., Stanković, S.: 'Robust Time-Frequency Distributions with Complex-lag Argument', EURASIP Journal on Advances in Signal Processing, 2010, Article ID 879874, 10 pages

[26] Donoho, D. L.: 'Compressed Sensing', IEEE Transactions on Information Theory, 2006, **52**, (4), pp. 1289-1306

[27] Tropp, J. A., Gilbert, A. C.: 'Signal Recovery From Random Measurements Via Orthogonal Matching Pursuit', IEEE Transaction on Information Theory, 2007, **53**, (12), pp. 4655-4666

[28] Stanković, S., Orović, I., Stanković, LJ.: 'An Automated Signal Reconstruction Method based on Analysis of Compressive Sensed Signals in Noisy Environment', Signal Processing, 2014, **104**, (/), pp. 43 – 50

[29] Stanković, S., Stanković, LJ., Orović, I.: 'A Relationship between the Robust Statistics Theory and Sparse Compressive Sensed Signals Reconstruction', IET Signal Processing, Special issue on Compressive Sensing and Robust Transforms, 2014, **8**, (3), pp. 223-229

[30] Zhang Y. D., Amin M. G.: 'Compressive sensing in nonstationary array processing using bilinear transforms', Proceedings of the Seventh IEEE Workshop on Sensor Array and Multi-Channel Signal Processing, Hoboken, NJ, June 2012, pp. 349-352

[31] Orović, I., Stanković, S., Thayaparan, T.: 'Time-Frequency Based Instantaneous Frequency Estimation of Sparse Signals from an Incomplete Set of Samples', IET Signal Processing, Special issue on Compressive Sensing and Robust Transforms, 2014, **8**, (3), pp. 239-245

[32] Omidvarnia, A., Azemi, G., O'Toole, J., Boashash, B.: 'Robust estimation of highly-varying nonlinear instantaneous frequency in monocomponent nonstationary signals', Signal Processing, 2013, **93**, (11), pp. 3251-3260

[33] Carrillo, R. E., Barner, K. E., Aysal, T. C.: 'Robust sampling and reconstruction methods for sparse signals in the presence of impulsive noise', IEEE Journal of Selected Topics in Signal Processing, 2010, **4**, (2), pp. 392–408

[34] Orović, I., Stanković, S., Amin, M.: 'Compressive Sensing for Sparse Time-Frequency Representation of Nonstationary Signals in the Presence of Impulsive Noise', SPIE Defense, Security and Sensing, Baltimore, Maryland, United States, May 2013, pp. 87170A-1 - 87170A-8

[35] Rauhut, H., Romberg, J., Tropp, J.: 'Restricted isometries for partial random circulant matrices', Applied and Computational Harmonic Analysis, 2012, **32**, (2), pp. 242–254

[36] Zhang, T.: 'Sparse recovery with orthogonal matching pursuit under RIP', IEEE Transactions on Information Theory, 2011, **57**, (9), pp. 6215–6221

[37] Yoon, Y., Amin, M.G.: 'Compressed sensing technique for high-resolution radar imaging', Proceedings of SPIE, Orlando, Florida, March 2008, 6968: 6968A-69681A-10

[38] Stanković, S., Orović, I., Amin, M.: 'Compressed Sensing Based Robust Time-Frequency Representation for Signals in Heavy-Tailed Noise', 11th International Conference on Information Sciences, Signal Processing and their Applications (ISSPA), July 2012, pp. 605-610

[39] Candes, E. J., Wakin, M. B., Boyd S.: 'Enhancing Sparsity by Reweighted l1 Minimization', Journal of Fourier Analysis and Applications, Special issue on sparsity, 2008, **14**, (5), pp. 877-905

[40] Blumensath, T., Davies, M. E.: 'Iterative Thresholding for Sparse Approximations', Journal of Fourier Analysis and Applications, 2008, **14**, (5-6), pp. 629-654

[41] Kunis S., Rauhut, H.: 'Random Sampling of Sparse Trigonometric Polynomials, II. Orthogonal Matching Pursuit versus Basis Pursuit', Journal Foundations of Computational Mathematics archive, 2008, **8**, (6), pp. 737-763

[42] Tropp, J., Needell, D.: 'CoSaMP: Iterative signal recovery from incomplete and inaccurate samples', Applied and Computational Harmonic Analysis, 2009, **26**, (3), pp. 301–321

[43] Figueiredo, M.A.T., Nowak, R.D., Wright, S.J.: 'Gradient Projection for Sparse Reconstruction: Application to Compressed Sensing and Other Inverse Problems', IEEE Journal of Selected Topics in Signal Processing, 2007, **1**, (4), pp. 586-597

[44] Jung, A., Taubock, G., Hlawatsch, F.: 'Compressive nonstationary spectral estimation using parsimonious random sampling of the ambiguity function', IEEE/SP 15th Workshop on Statistical Signal Processing, SSP '09., Cardiff, Wales, August 2009-September 2009, pp. 642-645

[45] Stanković, LJ., Stanković, S., Amin, M.: 'Missing Samples Analysis in Signals for Applications to L-estimation and Compressive Sensing', Signal Processing, 2014, 94, (/), pp. 401-408

[46] Gemmeke, J. F., Van Hamme, H., Cranen, B., Boves, L.: 'Compressive Sensing for Missing Data Imputation in Noise Robust Speech Recognition', IEEE Journal of Selected Topics in Signal Processing, 2010, **4**, (2), pp. 272-287