

# SIGNAL CHARACTERIZATION USING GENERALIZED “TIME-PHASE DERIVATIVES” REPRESENTATION

Bertrand Gottin<sup>†</sup>, Irena Orovic<sup>‡</sup>, Cornel Ioana<sup>†</sup>, Srdjan Stankovic<sup>‡</sup> and Jocelyn Chanussot<sup>†</sup>

<sup>†</sup>GIPSA-Lab, Grenoble INP, Domaine Universitaire, Saint-Martin d’Heres - FRANCE

E-mails : [bertrand.gottin,cornel.ioana,jocelyn.chanussot]@gipsa-lab.inpg.fr

<sup>‡</sup>University of Montenegro, Podgorica - MONTENEGRO

E-mails : [irenao,srdjan]@cg.ac.yu

## ABSTRACT

Natural signals are often characterized by a complex time-frequency behaviour. These signals exist in many different applications and systems from underwater acoustic to audio signals with sound attacks or electrical systems with partial discharges and commutation switches, for example. There is a huge number of Time-Frequency (TF) methods that aim to characterize these signals in terms of first phase derivative analysis (i.e Instantaneous Frequency Law). Recently, we introduced the time frequency distribution based on complex lag arguments. This distribution is able to reduce inner interferences terms which appear when studying non-linear TF components. It also offers access to an instantaneous law representation of any phase derivative order. In this paper, we use these two properties to study highly non-stationary signals as well as transient signals.

**Index Terms**— “Time-phase derivatives” representation, complex arguments, phase analysis

## 1. INTRODUCTION

Analysing signals characterized by a complex time-frequency behaviour is very challenging, due to the richness of the information described by the analyzed phenomena. In a large number of applications, the analysis of the time-frequency (TF) content provides an efficient solution for the characterization of diverse physical phenomena. Wave propagation through time-varying dispersive channels, micro-Doppler effects or mechanical signals are just three examples requiring an efficient time-frequency analysis of signals arising from these applications [1]. The signals associated to these applications are generally characterized by many non-linear time-frequency structures. An efficient analysis of such signal should highlight the time-frequency energy of signal structures despite of artefacts that inherently appear when using time-frequency representations (TFR). Hence, in the case of linear TFRs, the well-known trade-off between time and frequency resolutions has to be considered. This topic has been subject to a large number of works. An alternative to linear TFR is the concept

of quadratic TFRs [2]. One of the major research directions concerns the interference control in order to focus on time-frequency components of the signals. There are two types of interferences: inner interferences, generated in the case of non-linear time-frequency components and cross-terms, generated by the multi-component structures. The inner interferences type is usually addressed by non-linear TFR designed with help of time or frequency warping concept [1], [3].

Recently, the complex time distribution concept has been introduced in [4] as an efficient way to produce almost completely concentrated representations along the polynomial instantaneous frequency laws (IFL) of order 4 or less. In [5] we propose the generalization of the complex time distribution producing, in the mono component case, highly concentrated distributions around arbitrary non-linear IFLs. In this paper we will show that the complex time distribution concept is able to deal with transient signals characterization. This interest of processing in transient signal context is achieved via the derivability property of this distribution.

The paper is organized as follows. In Section 2 a presentation of the complex time distribution concept is done. The capability of the generalized version of this concept to deal with transient signals is presented in Section 3. We conclude in Section 4.

## 2. TIME-FREQUENCY DISTRIBUTION BASED ON COMPLEX LAG ARGUMENTS

The concept of complex lag distributions has been introduced in [4] as a way for inner interferences reduction with respect of Wigner distribution. Recently, this concept has been generalized in order to focus on arbitrary instantaneous phase derivative of a signal [5]. Let us consider the signal defined as:

$$s(t) = A \cdot e^{j\phi(t)} \quad (1)$$

The case of  $A$  depending of  $t$  can also be addressed since the effect of slowly varying amplitude is “visible” on the instantaneous phase. Otherwise, after a signal normalization, we can consider  $A = 1$ .

In order to better understand the concept of complex lag

distribution and its generalization, applied on such a signal, let us introduce the very well known Wigner distribution with appropriate analysis of its moment and lags definition.

### 2.1. The Wigner Ville Distribution

The Wigner Ville distribution of a signal  $s(t)$  is by definition [2]:

$$WVD(t, \omega) = \mathfrak{F}_\tau \left[ \overbrace{s\left(t + \frac{\tau}{2}\right) s^*\left(t - \frac{\tau}{2}\right)}^{M_{wv}(t, \tau)} \right] \quad (2)$$

This corresponds to the Fourier transform, with respect of the lag variable  $\tau$ , of a higher order moment denoted  $M_{wv}(t, \tau)$ . As illustrated in Fig. 1, this moment is calculated using two lag coefficients taken on the real axis.

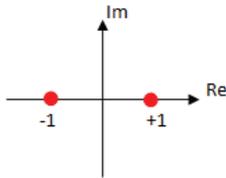


Fig. 1. Lag coefficients taken on the real axis

For a signal defined in (1), the expression of the moment becomes:

$$M_{wv}(t, \tau) = e^{j[\phi(t+\frac{\tau}{2})-\phi(t-\frac{\tau}{2})]} \quad (3)$$

Let us express the signal phase law in terms of Taylor series expansion:

$$\phi\left(t + \frac{\tau}{2}\right) = +\phi'(t) \frac{\tau}{2^1 1!} + \phi^{(2)}(t) \frac{\tau^2}{2^2 2!} + \phi^{(3)}(t) \frac{\tau^3}{2^3 3!} + \dots$$

$$\phi\left(t - \frac{\tau}{2}\right) = -\phi'(t) \frac{\tau}{2^1 1!} + \phi^{(2)}(t) \frac{\tau^2}{2^2 2!} - \phi^{(3)}(t) \frac{\tau^3}{2^3 3!} + \dots$$

Using the derivation results above, the expression (3) becomes:

$$M_{wv}(t, \tau) = e^{j\phi'(t)\tau} \times e^{j[\phi^{(3)}(t) \frac{\tau^3}{2^2 3!} + \dots]} \quad (4)$$

By substituting (4) in (2), we obtain a new analytical expression (5) of the WVD defining it as an ideally concentrated representation of the IFL but degraded because of the convolution with a spreading factor.

$$WVD(t, \omega) = \delta\left(\omega - \phi'(t)\right) *_{\omega} \mathfrak{F}_\tau \left[ e^{jQ_{wv}(t, \tau)} \right] \quad (5)$$

where  $Q_{wv}$  is the spread function defined as:

$$Q_{wv}(t, \tau) = \phi^{(3)}(t) \frac{\tau^3}{2^2 3!} + \phi^{(5)}(t) \frac{\tau^5}{2^4 5!} + \phi^{(7)}(t) \frac{\tau^7}{2^6 7!} \dots$$

From this spread function expression, it is easy to understand that the concentration of the WV representation for a chirp signal (polynomial phase law of second order) will be optimal in so far as all  $\phi$ 's derivatives terms in  $Q_{wv}$  will be equal to zero.

### 2.2. The Complex-Time Distribution

The Complex-Time distribution of a signal  $s(t)$  is by definition [4]:

$$CTD(t, \omega) = \mathfrak{F}_\tau \left[ \overbrace{s\left(t + \frac{\tau}{4}\right) s^*\left(t - \frac{\tau}{4}\right) s^{-j}\left(t + j\frac{\tau}{4}\right) s^j\left(t - j\frac{\tau}{4}\right)}^{M_{ct}(t, \tau)} \right] \quad (6)$$

In the same way as WVD, this corresponds to the Fourier transform, with respect of the lag variable  $\tau$ , of a higher order moment denoted  $M_{ct}(t, \tau)$ . As illustrated in Fig. 2, this moment is in this case of order four and calculated using two lag coefficients taken on the real axis as well as on the imaginary axis, hence the concept of ‘‘complex-time arguments’’.

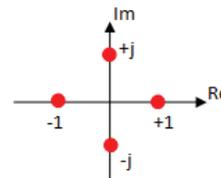


Fig. 2. Lag coefficients taken on the real and imaginary axis

Following the same frame of analysis described in subsection 2.1 leads to a new expression of  $CTD$  defined with the same form as (5). The spread function for this distribution is [4]:

$$Q_{ct}(t, \tau) = \phi^{(5)}(t) \frac{\tau^5}{4^4 5!} + \phi^{(9)}(t) \frac{\tau^9}{4^8 9!} + \phi^{(13)}(t) \frac{\tau^{13}}{4^{12} 13!} \dots$$

Defining a distribution using well-chosen ‘‘complex-lag’’ arguments ( $+j$  and  $-j$  on the imaginary axis) involves a significant decreasing of the spread factor. The first term of  $Q_{ct}(t, \tau)$  is of the fifth order. The terms of phase derivatives of order 3, 7, 11, ... are completely eliminated and all remaining terms are much more reduced with respect to the ones in the Wigner distribution. The Complex Time Distribution improves the concentration of the IFL representation comparing to the one obtained by Wigner Distribution (Fig. 4). In the case of a non-linear and rapidly varying TF structure, the inner interferences are strongly reduced.

### 2.3. The Generalized Complex-time Distribution

Recently, a generalization of the concept of CTD has been defined [5]. The starting point of this generalization procedure was the Cauchy's integral formula [6]. Using this theorem, it is possible to compute the  $K^{th}$  order derivative of the instantaneous phase as:

$$\phi^{(K)}(t) = \frac{K!}{2\pi j} \oint_{\gamma} \frac{\phi(z)}{(z-t)^{K+1}} dz \quad (7)$$

This relation shows the interest of the complex time concept: the  $K^{th}$  order derivate of function  $\phi$  at instant  $t$  can be computed as the complex integral over the integration path  $\gamma$  defined, in the complex plane, around this point. Applying the theory of Cauchy's integral theorem [6] and considering a circle as integration path, the expression (7) becomes [5]:

$$\phi^{(K)}(t) = \frac{K!}{2\pi\tau^K} \int_0^{2\pi} \phi(t + \tau e^{j\theta}) e^{-jK\theta} d\theta \quad (8)$$

As illustrated in Fig. 3, the discrete version of (8) is defined for  $\theta = 2\pi p/N$  and  $p = 0, \dots, N-1$ , where  $N$  is the number of discrete values of the angle  $\theta$  (expression (9)).

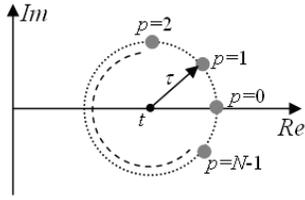


Fig. 3. The complex lag coordinates

$$\phi^{(K)}(t) = \frac{K!}{N\tau^K} \sum_{p=0}^{N-1} \phi\left(t + \tau e^{j\frac{2\pi p}{N}}\right) e^{-j\frac{2\pi p K}{N}} + \varepsilon \quad (9)$$

where  $\varepsilon$  is the discretization error.

Using the property of the unitary roots  $\omega_{N,p} = e^{j2\pi p/N}$  and the variable change  $\tau \leftarrow \sqrt[K]{\tau \frac{K!}{N}}$ , the expression (9) becomes:

$$\sum_{p=0}^{N-1} \phi\left(t + \omega_{N,p} \sqrt[K]{\tau \frac{K!}{N}}\right) \omega_{N,p}^{N-K} = \phi^{(K)}(t) \tau + Q(t, \tau) \quad (10)$$

where  $Q$  is the spread function defined as [5]:

$$Q(t, \tau) = N \sum_{r=1}^{+\infty} \phi^{(Nr+K)}(t) \frac{\tau^{\frac{Nr}{K}+1}}{(Nr+K)!} \left(\frac{K!}{N}\right)^{\frac{Nr}{K}+1} \quad (11)$$

As indicated by (10) and (11), the sum of the phase samples defined in the complex coordinates (left side of 10) is *linear* depending on  $\tau$  if the  $\phi$ 's derivatives of orders greater than  $N+K$  are 0. In order to exploit this property we define the generalized complex-lag moment (GCM) of  $s$  as the operation leading to (10):

$$\begin{aligned} GCM_N^K[s](t, \tau) &= \prod_{p=0}^{N-1} s^{\omega_{N,p}^{N-K}} \left(t + \omega_{N,p} \sqrt[K]{\tau \frac{K!}{N}}\right) \\ &= e^{j\phi^{(K)}(t)\tau + jQ(t, \tau)} \end{aligned}$$

The computation of GCMs implies the evaluation of signal samples at complex coordinates. This is achieved using the analytical continuation of a signal defined as [5]:

$$s(t + jm) = \int_{-\infty}^{+\infty} S(f) e^{-2\pi m f} e^{j2\pi f t} df$$

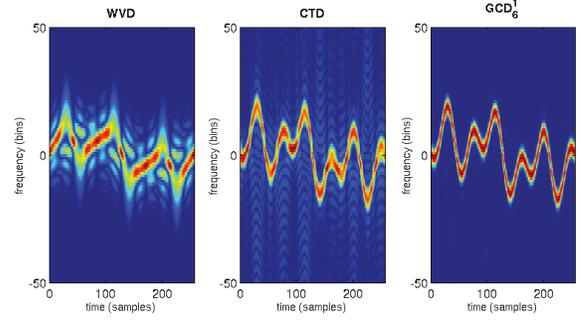


Fig. 4. Inner interferences reduction property of GCD

where  $S(f)$  is the Fourier transform of signal  $s$ . Taking the Fourier transform of GCM with respect to  $\tau$ , we define the generalized complex-lag distribution (GCD):

$$\begin{aligned} GCD_N^K[s](t, \omega) &= \mathfrak{F}_\tau [GCM_N^K[s](t, \tau)] \\ &= \delta\left(\omega - \phi^{(K)}(t)\right) *_{\omega} \mathfrak{F}_\tau \left[e^{jQ(t, \tau)}\right] \end{aligned} \quad (12)$$

As stated by this definition, the  $K^{th}$  order distribution of the signal, obtained for  $N$  complex-lags, highly concentrates the energy around the  $K^{th}$  order derivate of the phase law. This concentration is optimal if the  $\phi$ 's derivatives of orders greater than  $N+K$  are 0, exactly like in the case of chirps represented by Wigner distribution.

The general definition (12) leads to a large number of TFRs, part of them well known in literature. For example, for  $K=1; N=2$  the WVD is obtained (2.1) whereas the case  $K=1; N=4$  corresponds to the complex-time distribution (CTD) (2.2). In [5] we have shown that increasing the number of complex lags leads to an attenuation of inner interferences due to the time-frequency non-linearity. This is illustrated by the example in Fig. 4 for the following test signal:

$$s_1(t) = e^{j(3\cos(\pi t) + \frac{4}{3}\cos(3\pi t) + \frac{2}{3}\cos(6\pi t))}$$

We remark the better concentration of time-frequency energy in the case of  $GCD_6^1$  than in the case of the other TFRs. This is analytically proved by the spread function expression (11) and illustrated by the example Fig. 4.

The next example (Fig. 5) points out on the derivability property of GCD. We consider another highly non-stationary test signal defined as:

$$s_2(t) = e^{j(6\cos(\pi t) + \frac{2}{3}\cos(3\pi t) + \frac{4}{3}\cos(5\pi t))}$$

The Fig. 5 plots in the top the analytic derivatives of first, second and third orders of the IPL of this signal. We plot, in the bottom of the Fig. 5, the GCDs of the same orders. We remark that the theoretical derivatives are correctly represented by the GCDs of corresponding order, justifying the derivation property of the complex-lag distribution.

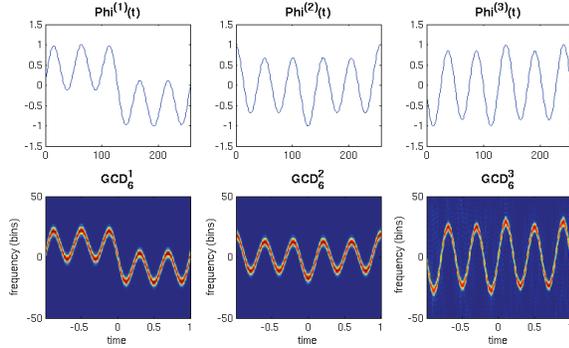


Fig. 5. Derivation of instantaneous phase by GCDs

### 3. APPLICATION TO TRANSIENT SIGNALS

The signal  $s$  used in this application is a train of three frequency modulations (FM) corrupted by some additive noise (SNR=35dB). The three FM have short duration (two linear FM in phase opposition on 128 samples and one parabolic FM on 64 samples) compared to the analysis time frame (1510 samples). Such signal could correspond to a received signal from two different radars using linear and parabolic FM waveforms, respectively (Fig. 6.a).

In this section, the derivability property of GCD is used in order to enable the characterization of transient natures which would be more difficult using just time-frequency representation because of confusing TF signatures. As shown in Fig. 6.b, the two linear FM have their well-expected TF structure whereas the transient parabolic FM looks like a chirp in the TF plane. This is because of the short duration effect on the large frame of analysis. We can observe that the linear and parabolic shapes of the FM are not easily distinguishable. The parabolic shape appear to be linear. Without a priori knowledge about the signal, the TF representation alone leads to considerate three transients of chirp nature which is actually wrong. To point out the true nature of the parabolic FM, the  $GCD_6^2$  is used. Considering the second order phase derivative lets to stationarize the linear FM and gives a linear signature for the parabolic FM (Fig. 6.c). The TF rate representation plane avoids the previous confusion.

### 4. CONCLUSION

The Generalized Complex-time Distribution offers the capability to focus on arbitrary derivatives of a signal phase law. This distribution is equivalent to a “Time-Phase derivatives” tool giving highly concentrated representation of the instantaneous phase derivative we are interested in. For signals with time-frequency structure characterized by non-linearity and quick variations, the GCD proves its efficiency. Having access to any phase derivative order makes possible useful applications such as the order reduction of a polynomial phase law or the removal of quasi-stationary corrupting signals in multi-components case [7]. This derivability property shows

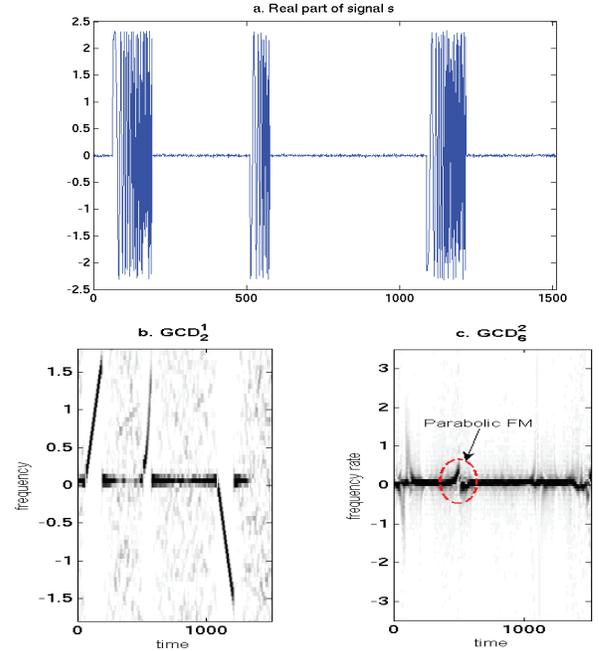


Fig. 6. (a) Signal  $s$  composed of two linear FM, one parabolic FM and additive noise (SNR=35dB); (b)  $GCD_2^1$  of  $s$ ; (c)  $GCD_6^2$  of  $s$

its efficiency also for transient signals characterization. Two transient informations of different nature but having similar-looking TF signatures can be characterized thanks to more specific signatures obtained from the representation of higher orders derivatives of the phase.

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