

Time-Frequency Rate Representation for IF Rate Estimation of Signals with Fast Varying Phase Function

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Abstract

An efficient approach for instantaneous frequency (IF) rate estimation is proposed. It is based on a time-frequency rate representation with complex-lag argument. The proposed representation is derived in a manner to reduce the spread factor, and thus to provide high concentration along the IF rate. Due to the significantly reduced spread factor, it is suitable for periodically frequency modulated signals whose phase vary fast even within a few samples. The theory is proven by the example.

Keywords - instantaneous frequency rate, time-frequency rate representation, complex-lag argument

1. Introduction

The instantaneous frequency rate is defined as the second derivative of the signal's phase function, i.e. the first derivative of the instantaneous frequency. For a frequency modulated signal $s(t) = Ae^{j\phi(t)}$, with a phase function $\phi(t)$, the IF rate is defined as:

$$\Omega(t) = \frac{d^2\phi(t)}{dt^2}. \quad (1)$$

Hence, an ideal representation of the IF rate can be written in the form:

$$TF_{IFR}(t, \Omega) = \delta(\Omega - \phi^{(2)}(t)) \quad (2)$$

where $\phi^{(2)}(t)$ denotes the second phase derivative, while δ is delta pulse. The IF rate estimation can be useful in many practical applications such as radar, sonar, communications and video surveillance. Namely, the IF rate can be used to estimate the time-varying acceleration of radar moving targets [1],[2]. Also, it can be used for

the estimation of instantaneous acceleration for object tracking in video surveillance applications [3],[4].

One of the commonly used techniques for the IF rate estimation is based on the time-frequency rate representation called the cubic phase function [5]. It is suitable for the IF rate analysis of the cubic phase signals. However, for higher order non-linearities of the phase function, this method cannot provide good concentration along the IF rate. An IF rate estimator with a second order non-linearity for high-order polynomial phase signals has been proposed in [6]- [8].

In this paper we propose a second order time-frequency rate representation for IF rate estimation of periodically frequency modulated signals with fast varying IF rate. The proposed representation is based on the complex-lag argument. Generally, the complex-lag time-frequency distributions has been introduced to provide high concentration along the IF for signals with fast varying phase function [9]-[11]. Therein, it has been shown that the complex-lag distribution outperforms commonly used time-frequency distributions such as the Wigner distribution. Thus, the complex-lag argument is considered also within the time-frequency rate representation in order to provide high concentration along the IF rate. The efficiency of the proposed representation is illustrated by the examples.

The paper is organized as follows. The theoretical background on the IF rate estimation is given in Section II. The time-frequency rate representation based on the complex-lag argument is proposed in Section III. The experimental results are presented in Section IV, while the concluding remarks are given in Section V.

2. Theoretical background

The cubic phase function, as one of the commonly used time-frequency rate representations, is defined as [5]:

$$CP(t, \Omega) = \int_0^{\infty} s(t+\tau)s(t-\tau)e^{-j\Omega\tau^2} d\tau. \quad (3)$$

Thus, for a cubic phase signal:

$$s(t) = Ae^{j(a_0 + a_1t + a_2t^2 + a_3t^3)}, \quad (4)$$

the function $CP(t, \Omega)$ produces peaks along the IF rate law $\Omega = 2(a_2 + 3a_3t)$:

$$CP(t, \Omega) = A^2 e^{j2(a_0 + a_1t + a_2t^2 + a_3t^3)} \times \int_0^\infty e^{j(2(a_2 + a_3t) - \Omega)\tau^2} d\tau. \quad (5)$$

Consider the signal moment: $M(t) = s(t + \tau)s(t - \tau)$, where $s(t) = Ae^{j\phi(t)}$. The Taylor series expansion applied to the phase of $M(t)$ results in:

$$\begin{aligned} \phi_M(t, \tau) &= \phi(t + \tau) + \phi(t - \tau) = \\ &= 2\phi(t) + 2\frac{\phi^{(2)}(t)\tau^2}{2!} + 2\frac{\phi^{(4)}(t)\tau^4}{4!} + 2\frac{\phi^{(6)}(t)\tau^6}{6!} + \dots \end{aligned} \quad (6)$$

All phase derivatives apart from the second one represent the spread factor (the fourth and higher order phase derivatives). It defines spreading of the time-frequency rate concentration. According to (6), the cubic phase function can be efficiently used for signals whose phase has derivatives up to the third order. For phase function with higher non-stationarity, the influence of higher order phase derivatives becomes significant. These terms reduce the concentration within the time-frequency rate domain, which further influence the precision of the IF rate estimation.

3. Introducing Highly Concentrated Time-Frequency Rate Representation with a Complex-lag Argument

The complex-lag argument has been introduced into the definition of time-frequency distributions for the purpose of fast varying IF estimation [9],[11]. Since the signal is available along the real axis only, the signal terms with complex-lag argument has to be calculated by using signal with a real argument.

The complex-lag distributions are able to provide high concentration along the IF. An appropriate combination of signal terms with complex-lag argument provides elimination of certain higher phase derivatives from the spread factor. In this way, the spread factor decreases,

providing high concentration in the time-frequency plane. This is especially emphasized for signals with highly non-stationary IF, when the complex-lag distributions provides significantly better results comparing to the Wigner distribution, for example. This concept can be used not only for the IF, but also for the IF rate representation.

Consequently, in order to improve the concentration along the IF rate, we propose the complex-lag time-frequency rate representation. Let us start from the time-frequency rate representation given in the form:

$$CTFR(t, \Omega) = \int_0^\infty s(t + \sqrt{j}\tau)s(t - \sqrt{j}\tau)e^{-j\Omega\tau^2} d\tau, \quad (7)$$

where $j = \sqrt{-1}$. Note that the proposed time-frequency rate representation is of the second order like the cubic phase function.

In [11], it has been shown that the signal with complex-lag argument can be calculated as follows:

$$s(t \pm j\tau) = \frac{1}{2\pi} \int_{-\infty}^\infty S(\omega)e^{\mp\omega\tau}e^{j\omega t} d\omega, \quad (8)$$

where $S(\omega)$ is the Fourier transform of signal $s(t)$. In other words, the signal with complex-lag argument is calculated as the inverse Fourier transform of $S(\omega)e^{\mp\omega\tau}$. Note that the real exponential functions $e^{\mp\omega\tau}$, for large values of $\omega\tau$ could exceed the computer precision range, producing calculation errors. Thus, in the direct numerical implementations the above relation should be carefully used.

Accordingly, the signal term with complex-lag argument used in the definition of $CTFR(t, \Omega)$ can be calculated as:

$$s(t \pm \sqrt{j}\tau) = \frac{1}{2\pi} \int_{-\infty}^\infty S(\omega)e^{j\omega \left(t \pm \left(\frac{\sqrt{2}}{2} + j\frac{\sqrt{2}}{2} \right) \tau \right)} d\omega. \quad (9)$$

In order to obtain the final form of the time-frequency rate representation defined by (7), let us observe the phase of the signal moment $M(t, \tau) = s(t + \sqrt{j}\tau)s(t - \sqrt{j}\tau)$. The Taylor series expansion of the moment phase function is given by:

$$\phi_M(t, \tau) = 2\phi(t) + j\phi^{(2)}(t)\tau^2 - 2\frac{\phi^{(4)}(t)\tau^4}{4!} \dots \quad (10)$$

Thus, all odd phase derivatives are eliminated. In order to focus on the second order derivative and to eliminate the influence of the fourth order derivative for example, we introduce the following modification:

$$M_a(t, \tau) = e^{j \log |M(t, \tau)|}. \quad (11)$$

Therefore, the highly concentrated time-frequency rate representation is obtained as:

$$CTFR(t, \Omega) = \int_0^{\infty} M_a(t, \tau) e^{-j\Omega\tau^2} d\tau. \quad (12)$$

The spread factor of $CTFR(t, \Omega)$ can be now written as:

$$Q(t, \tau) = -2 \frac{\phi^{(6)}(t)\tau^6}{6!} + 2 \frac{\phi^{(10)}(t)\tau^{10}}{10!} - 2 \frac{\phi^{(14)}(t)\tau^{14}}{14!} \dots$$

Hence, the first term in the spread factor is of the sixth order, while the total number of terms is significantly reduced comparing to the cubic phase function. Namely, the spread factor contains only the phase derivatives in the form $4n+2$, $n=1,2,\dots$. Consequently, the proposed time-frequency rate representation will provide significantly higher concentration of the fast varying IF rate.

4. Numerical Example

In this Section, the efficiency of the proposed time-frequency rate representation is experimentally demonstrated. For this purpose, a periodically frequency modulated signal, with fast varying phase function is considered. The signal is given in the form [11]:

$$s(t) = e^{j(6\cos(\pi t) + 2/3\cos(3\pi t) + 2/3\cos(5\pi t))}.$$

In real cases, this kind of signal corresponds, for example, to radar signal produced by non-uniform rotation of reflecting point. The exact IF rate is given by:

$$\Omega(t) = -6\pi^2\cos(\pi t) - 6\pi^2\cos(3\pi t) - 50/3\pi^2\cos(5\pi t).$$

The signal is calculated for $t=-1:\Delta t:1-\Delta t$, where $\Delta t = 2/N$, while $N=128$. The proposed time-frequency rate representation is illustrated in Figure 1.a. In order to compare the results, the cubic phase function is calculated and shown in Figure 1.b. Note that for the considered IF rate, the cubic phase function does not provide satisfactory concentration and cannot properly follow the IF rate variations.

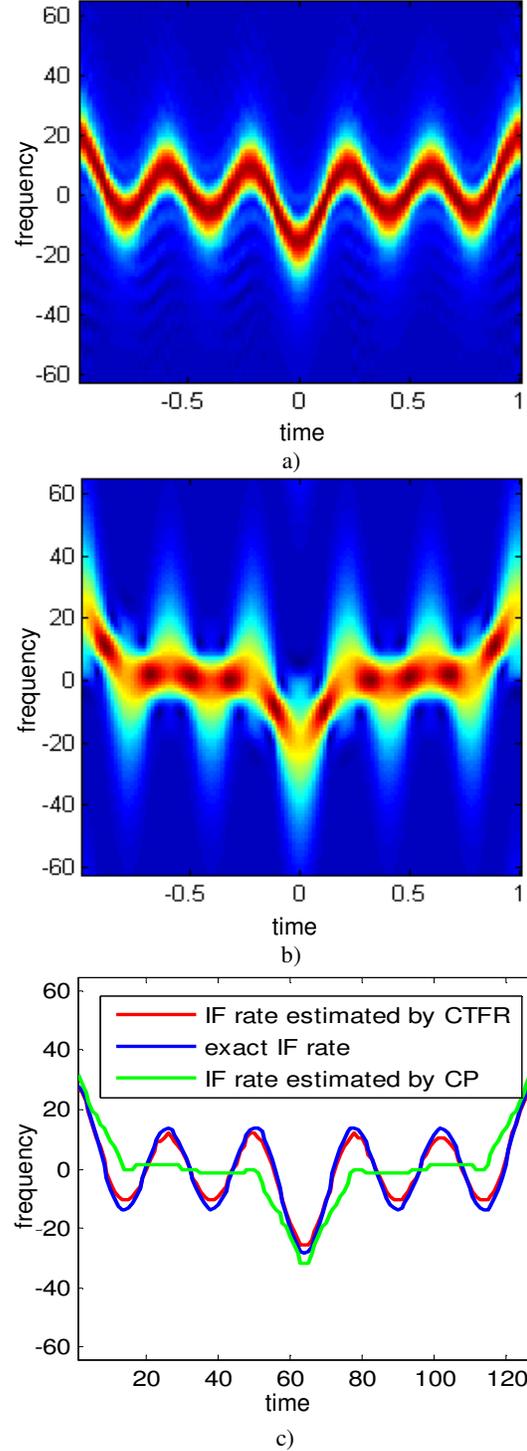


Figure 1. a) The results for the proposed time-frequency rate representation CTFR, b) the results for the cubic phase function, c) the IF rate estimated by using the proposed CTFR and by using the cubic phase function

On the other hand, the proposed representation, provide good concentration in the time-frequency rate domain, and thus higher precision of the IF rate estimation (Figure 1.c).

The precision of the IF rate estimation is expressed in terms of the mean square error as follows:

$$MSE = \frac{1}{N} \sum (\Omega(t) - \hat{\Omega}(t))^2, \quad (13)$$

where $\Omega(t)$ denotes the exact IF rate, while the estimated $\hat{\Omega}(t)$ is: $\hat{\Omega}(t) = \arg \max_{\omega} \{TFR(t, \Omega)\}$, TFR is time-frequency rate representation. The MSEs for the cubic phase function CP and for the proposed time-frequency rate representation CTFR are given in Table 1.

Table 1. Comparison of IF rate estimation in terms of MSE

Time-frequency rate representation (TFR)	Cubic Phase function (CP)	Proposed CTFR
MSE	99.33	4

5. Conclusion

A time-frequency rate representation that provides high concentration along the IF rate is proposed. It is based on the use of the second order signal moment with complex-lag argument. The proposed representation provides significant reducing of the spread factor comparing to the existing cubic phase function. Namely, the number of higher phase derivatives that cause concentration spreading is reduced twice and contains only the derivatives in the form $4n+1$. Therefore, the proposed distribution can be efficiently used for the IF rate estimation of signals with fast varying phase function, as it is demonstrated by the example. The results have shown that the proposed representation provides high precision of the IF rate estimation.

6. References

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