A Robust Form of the Ambiguity function

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Abstract **-A form of the Ambiguity function appropriate for signals affected by mixture of the Gaussian and impulsive noise is introduced. Then, it is used for calculation of the robust distributions belonging to the Cohen class. The proposed class of robust distributions may be used for time-frequency analysis and IF estimation of a wide class of signals corrupted by a heavy tailed noise. The theory is justified by a set of numerical examples.**

Key words **— Ambiguity function, Cohen class of distributions, Robust time-frequency analysis**

I. INTRODUCTION

TIME-frequency analysis is used for signals with time varying spectar. Namely, in that case, because of non stationary nature, the Fourier transform can not provide enough information about the signal [1], [2]. It is important to emphasize that no time-frequency distribution provides the optimal solution for all types of signals. It is the reason why a number of time-frequency distributions are defined in order to meet requirements for various specific cases. The simplest and commonly used distribution is the Spectrogram. However, it lacks of a poor time-frequency resolution for signals that are not constantly modulated. The resolution can be improved using the Wigner distribution, but in the case of multicomponent signals it introduces the cross terms. Various quadratic distributions belonging to the Cohen class are introduced in order to supress or reduce the cross terms (e.g. Choi Willams, Zhao Atlas Marks,...). These distributions are used in many practical applications such as: radar signal processing, biomedical signals, seismics signals, digital watermarking etc [1]-[9].

The Cohen class of distributions can be defined using the Ambiguity function . Note that the Ambiguity function is not only important for constructing the time-frequency distributions belonging to the Cohen class, but also it is commonly used in radar signal processing [10]. Here, we aim to provide robust Ambiguity function which can be used by itself and maybe as a base for the robust Cohen class. Namely, this class represents two dimensional Fourier transform of the Ambiguity function filtered by a certain two dimensional function, called kernel. The cross terms in the time-frequency domain can be significantly reduced by an appropriate selection of the kernel function. The kernel is a two dimensional low pass filtering function. It is designed in the ambiguity domain taking into account the property that the cross terms are dislocated from the origin, so a low frequency filtering function can supress or reduce them. Moreover, the kernel function specifies a type of distribution and its properties.

Consider a discrete form of the Cohen class that can be used in practical applications:

$$CD(l,k) = \sum_{m=-\frac{Np}{2}}^{\frac{Np}{2}} \sum_{p=-\frac{N}{2}}^{\frac{N}{2}-1} c(p,m)A(p,m)e^{-j\frac{2\pi}{N}pl-j\frac{2\pi}{N_p}km}, \quad (1)$$

The kernel function is denoted by $c(p,m)$, while $A(p,m)$ represents the Ambiguity function:

$$A(p,m) = \sum_{n=-\frac{N}{2}}^{\frac{N}{2}-1} x(n+m)x^*(n-m)e^{-j\frac{2\pi}{N}pn}. \quad (2)$$

The total number of samples is N, while Np+1 is the number of samples within the rectangular window (used for the Ambiguity function calculation).

In real applications we deal with signals corrupted by noise. Note that the standard form of the Cohen class is optimal for Gaussian noise only. However, noises with other probability density function (pdf) characteristics are often present. Namely, impulsive noise or a combination of impulsive and Gaussian noise occures very often in practical applications. In these cases a robust form of distributions, based on the median or L statistics, should be used [11]-[14]. Even though the usage of L statistics can increase the calculation complexity, efficient hardware solutions exist [15].

In this paper, based on the L statistics, we introduce a robust form of the Ambiguity function. It has been also used for defining a robust form of the Cohen class of distributions. This robust form is obtained directly in the ambiguity domain, in difference with the other distribution such as Spectrogram and Wigner distribution where the robust forms are calculated in the time-frequency domain.

The paper is structured as follow. After an introduction, a basic of the Huber estimation theory is described in the second section, where the mean, median and L forms of estimators are given as well. A robust form of the Ambiguity function is proposed in the thirdth section, while the numerical examples are presented and discussed in the section four. The conclusion is given within the

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section five.

II. Theoretical background

Based on the Huber estimation theory robust statistics approaches have been introduced. The aim is to obtain a signal estimator as a solution of the optimization problem that depends on characteristics of noise . Let us consider a discrete signal $f(n)$ corrupted by noise:

$$f(n) = s(n) + v(n), \tag{3}$$

where $s(n)$ is the signal without noise, while $v(n)$ is a noise. The Hubers signal estimator can be obtained by solving the following optimization problem:

$$x(n) = \mu = \arg\min_{\mu} \sum_{k=n-N}^{n+N} F\left[f(k) - \mu\right]. \tag{4}$$

Thus, $2N+1$ signal samples of $f(n)$ are used for $x(n)$ estimation. The loss function is denoted by $F(e)$ where $e = f(k) - \mu$. The function F depends on noise pdf and it is defined as:

$$F(e) = -\log p_v(e). \tag{5}$$

Note that the thermal noise can be modeled by Gaussian distribution, so the loss function is:

$$F(e) = -\log p(e) = -\log\left(\frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-(e-\mu)^2}{2\sigma^2}}\right) \sim e^2. \tag{6}$$

By solving the optimization problem the mean form of the estimator follows:

$$\mu = \frac{1}{2N+1} \sum_{k=n-N}^{n+N} f(k) = \underset{k\in[n-N,n+N]}{mean}\left\{f(k)\right\} = x(n). \tag{7}$$

Similarly, the loss function for an impulsive noise (which can be described with Laplacian distribution) is:

$$F(e) = -\log p(e) = -\log\left(\frac{1}{2b} e^{\frac{-|e-\mu|}{b}}\right) \sim |e|. \tag{8}$$

Replacing (8) in (4) and solving the minimization problem, the median form of estimator is obtained:

$$-\sum_{k=n-N}^{n+N} sign\left[f(k) - \mu\right] = 0 \rightarrow \sum_{k=n-N}^{n+N} sign\left[f(k) - \mu\right] = 0. \tag{9}$$

It is important to note that the previous approach is very sensitive on the pdf form. Thus, in the case of a mixture of the Gaussian and impulsive noise the main difficulty is to determine a noise pdf. It is the reason why the L estimation is introduced. Here, the signal estimator is defined as:

$$x(n) = \sum_{i=1}^{N} a_i f_i(n), \tag{10}$$

where the coefficients a_i satisfy the following conditions:

$$\sum_{i=1}^{N} a_i = 1, \qquad a_i = a_{N-i}. \tag{11}$$

The most commonly used form is α trimmed form of L-estimator. This estimator uses a sorted sequence and it selects a few samples around the median. Then the mean value is calculated for them, (the remaining samples are multiplied by zero). This form of the estimator can be applied either in signal or transform domain.

III. Robust Ambiguity function

In this section the median and L form of the Ambiguity function will be defined. Unlike the already defined median and L form of estimators based on the Spectrogram or Wigner distribution the optimization problem has to be solved in the Ambiguity domain. Therefore, we use Huber statistics and solve the optimization method in the Ambiguity domain.

A. Median form of the Ambiguity function

The median form of the Ambiguity function can be obtained by solving the following problem:

$$A_M(p,m) = \arg\min_{\mu} \sum_{n=-N/2}^{N/2} F(e). \tag{12}$$

If we take $F(e) = |e|$, where e is:

$$e = x(n+m)x^*(n-m)e^{-j2\pi np/N} - \mu, \tag{13}$$

the optimization problem has the form:

$$A_M(p,m) = \arg\min_{\mu} \underbrace{\sum_{n=-N/2}^{N/2} |e|}_{J}. \tag{14}$$

The solving procedure includes the following equations:

$$\frac{\partial J}{\partial \mu} = 0 \Rightarrow$$

$$-\sum_{n=-N/2}^{N/2} sign\left\{x(n+m)x^*(n-m)e^{-j2\pi np/N} - \mu\right\} = 0,$$

$$\sum_{n=-N/2}^{N/2} \frac{x(n+m)x^*(n-m)e^{-j2\pi np/N} - \mu}{\left|x(n+m)x^*(n-m)e^{-j2\pi np/N} - \mu\right|} = 0.$$

The solution is obtained as μ:

$$\begin{aligned}\mu \sum_{n=-N/2}^{N/2} \frac{1}{\left|x(n+m)x^*(n-m)e^{-j2\pi np/N} - \mu\right|} &= \\ = \sum_{n=-N/2}^{N/2} \frac{x(n+m)x^*(n-m)e^{-j2\pi np/N}}{\left|x(n+m)x^*(n-m)e^{-j2\pi np/N} - \mu\right|}&.\end{aligned} \tag{15}$$

Thus, the median form of the Ambiguity function (calculated in the ambiguity domain) is:

$$\begin{aligned}A_M(p,m) = \frac{1}{\sum_{n=-N/2}^{N/2} \frac{1}{\left|x(n+m)x^*(n-m)e^{-j2\pi np/N} - \mu\right|}} \text{ x}\\ \text{x} \sum_{n=-N/2}^{N/2} \frac{x(n+m)x^*(n-m)e^{-j2\pi np/N}}{\left|x(n+m)x^*(n-m)e^{-j2\pi np/N} - \mu\right|}.\end{aligned} \tag{16}$$

B. L-form of the Ambiguity function

The Ambiguity L estimator in α trimmed form is:

$$A_L(p,m) = \sum_{i=-N/2}^{N/2-1} a_i\left(r_i(p,m) + j\cdot i_i(p,m)\right) \tag{17}$$

where $r_i(p,m)$ i $i_i(p,m)$ are elements of the sorted sequences $R(p,m)$ i $I(p,m)$. Note that:

$$r_i(p,m) \in R(p,m),$$
$$R(p,m) = \left\{ \mathrm{Re}\left\{ x(n+m)x^*(n-m)e^{-j2\pi np/N} \right\}, n \in \left[-\frac{N}{2}, \frac{N}{2} \right) \right\}$$
$$i_i(p,m) \in I(p,m),$$
$$I(p,m) = \left\{ \mathrm{Im}\left\{ x(n+m)x^*(n-m)e^{-j2\pi np/N} \right\}, n \in \left[-\frac{N}{2}, \frac{N}{2} \right) \right\}. \tag{18}$$

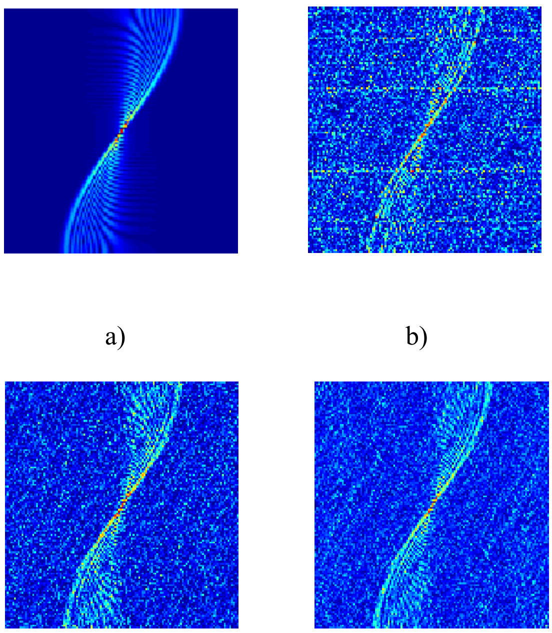
The coefficients a_i are defined as:

$$a_i = \begin{cases} \dfrac{1}{N(1-2\alpha)+4\alpha}, & for\ \ i \in \left[(N-2)\alpha, \alpha(2-N)+N-1 \right], \\ 0, & \text{otherwise.} \end{cases} \tag{19}$$

If we take $\alpha = 0$ the standard Ambiguity function follows, while for $\alpha = 0.5$ the median form is obtained.

IV. Numerical Results

Example 1: Consider the monocomponent signal $x(n) = e^{j32\cos(2\pi n)}$ affected by a mixed Gaussian and impulsive noise. The standard Ambiguty function of a non noisy signal is presented in Fig.1a. In Fig.1.b,c,d the standard, median and L form of the Ambiguity function, calculated for the noisy signal, are shown. The window of 128 samples width is used, while the total length of the signal is 256 samples.



a) b)

c) d)

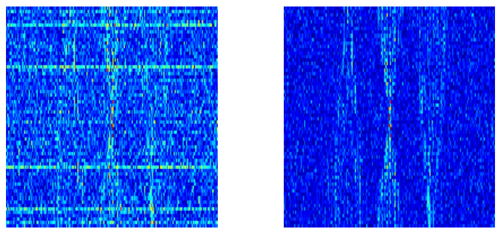
Fig. 1. Non noisy case: a) Ambiguity function; Noisy case: b) Ambiguity function of the noisy signal c) Median form of the Ambiguity function d) L-form of the Ambiguity function

Note that the standard form does not produce good results in the case of noisy signal. The representation is improved by the median form. As it is expected the L form provides the best results. This is also shown numerically in Table 1 by calculating the mean square error (MSE) for the obtained ambiguity functions.

Table 1: MSE for different ambiguity function estimators.

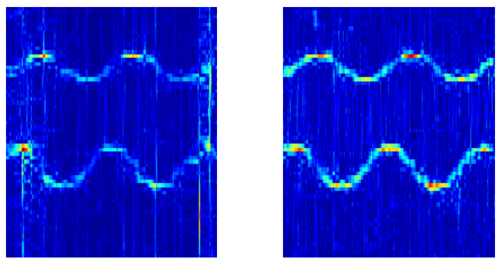
Ambiguity function	*MSE*
Standard form	4.08
Median form	3.76
L form	2.89

Example 2: In this example the multicomponent signal is considered. The signal consists of two sinusoidal frequency modulated signals corrupted by a mixture of the Gaussian and impulsive noise. The standard and L form of the Ambiguity functions are presented in Fig.2 a,b. Moreover the Choi Williams distribution is calculated based on these forms of the Ambiguity function. Obviously the time frequency representation obtained by using L form outperforms the standard one which contains errors that can be interpreted as signal components (Fig3,a,b).



a) b)

Fig. 2. a) Standard Ambiguity function of a noisy signal b) L-form of the Ambiguity function

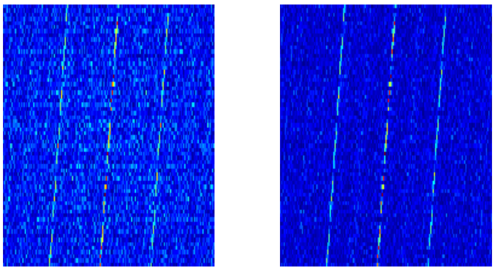


a) b)

Fig. 3. Choi-Williams distribution calculated using: a) Standard Ambiguity function b) L-form of the Ambiguity function.

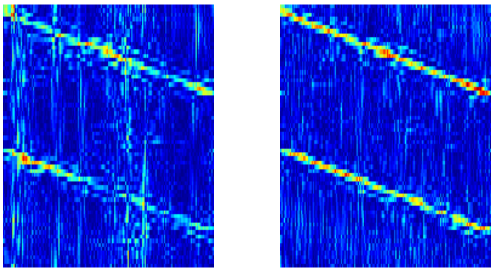
Example 3: A higher amount of the mixed noise is consider in this example. The signal consists of two linear frequency modlulated components. The standard Ambiguity function is given in Fig. 4a, while the L form is shown in Fig.4b. For both of them the Choi Williams distribution is calculated and shown in Fig. 5a and b.

As in the previous examples the L estimate based form provides better results than the standard appoach. Namely, some noisy components in the standrd form are as strong as the signal components, while they are significantly reduced by using the L form.



a) b)

Fig. 4. a) Standard Ambiguity function of a noisy multicomponent signal (mixture of the Gaussian and impulsive noise) b) L-form of the Ambiguity function



a) b)

Fig. 5. Choi-Williams distribution based on a) Standard Ambiguity function b) L-form of the Ambiguity function.

V. Conclusion

The median and L form of the Ambiguity function are proposed and considered in the paper. They follow as solutions of the optimization problem in the Ambiguity domain. The median form can be used for pure impulsive noise, modeled by Laplacian distribution. If the noise is a mixture of the Gaussian and Laplacian noise then the L form is more appropriate, as it is proven by numerical examples. Having in mind that the Ambiguity function is commonly used for the Cohen class distribution calculation, the counterpart of time frequency distributions are considered. The kernel that specifies Choi Williams distributions is used in the numerical examples. It is shown that the best time frequency representation is achived by using the L forms. A similar behavior is obtained for other distributions from the Cohen class.

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Rezime

U radu su predlozene nove forme Ambiguity funkcija, pogodne za analizu signala zahvaćenih mješavinom Gausovog i impulsnog suma. Predložene forme su korišćene za definisanje robusnih formi Cohen-ove klase dsitribucija, računatih preko ambiguity domena. Pokazuje se da su nove forme znatno optimalnijeod standardnih za analizu signala zahvaćenih mješovitim šumom.

ROBUSNE AMBIGUITY FUNKCIJE

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