A comparison of CS Reconstruction Algorithms for Multicomponent Nonlinear Phase Signals

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Abstract — **This paper analyzes the performance of different compressive sensing algorithms applied to signals with polynomial and cosine modulated phases, that usually appear in radar communications. In order to provide sparsity in the Fourier transform domain, the signal components are firstly demodulated by a direct parameter search method. In this way, the signals are sparsified in the DFT domain. The performance of reconstruction algorithms are compared using small number of measurements. The results are illustrated on examples.**

Keywords- Compressive sensing, signal reconstruction algorithms, radars, sparsity.

I. INTRODUCTION

In radar communications, the signal is transmitted to a target and reflected back to the radar. The information about desired characteristics of the target is extracted from reflected signal using proper processing techniques [1]-[3]. When transmitted signal reflects from the moving target, the carrier frequency is usually shifted due to the target motion. Also, vibration or rotation of target parts may induce additional frequency modulations on the returned signal which generate sidebands about shifted carrier – micro Doppler effect. For instance, the rigid body components could have the polynomial phase modulation, while the micro-Doppler resulting from the fast rotating parts can be modeled by the cosine modulation.

Nowadays, one of the main efforts in digital signal acquisition has been made toward the Compressive sensing (CS) approach allowing to reconstruct signals with much fewer measurements than it is done in traditional approaches [4]-[9]. Compressive sensing implies non-adaptive, linear projection of signal that preserves its structure. The necessary conditions for CS are the signal sparsity in certain basis and incoherence between measurement matrix and sparsity basis. Signal is reconstructed based on linear measurements using different optimization algorithms [4],[5],[10]-[14].

In this paper we analyze the possibility to use some existing CS algorithms for the reconstruction of signals with different phase modulations. Particularly, we observe the signal that consists of two different types of component: 1) components with polynomial phase that models rigid body, 2) cosine phase modulated component that models micro-Doppler effects. Nevertheless, one important step that needs to be included in the CS solvers is the component demodulation procedure [15]. Namely, the sparsity is ensured by applying the demodulation of components based on the detection of the phase parameters. Due to the multicomponent signal nature, the demodulation results in one main component, and certain interfering components. This feature may influence the reconstruction performance, and the idea is to analyze which algorithm provides optimal results in this circumstances.

The paper consists of four sections. The problem formulation and demodulation technique is given in Section II The review of some common CS reconstruction algorithms is given in Section III. Performance analysis of CS algorithms applied to signals with nonlinear phase modulation is given in Section IV. Section V is the conclusion.

II. COMPRESSIVE SENSING RECONSTRUCTION OF SIGNALS WITH NONLINEAR PHASE MODULATION

A. *Compressive sensing theory*

Observe a signal in R^n represented by using certain basis vectors $\{\psi_i\}_{i=1}^{N}$:

$$x = \sum_{i=1}^{N} \alpha_i \psi_i = \alpha\psi\,, \tag{1}$$

where α is $N\times 1$ vector of weighting coefficients. It is assumed that s is the number of non-zero coefficients in α. The signal x is compressible (and s-sparse) if its representation has just a few coefficients and many coefficients that can be approximated with zero. Samples of signal x are in fact linear measurements that can be obtained as follows:

$$y=\phi x=\phi\psi\alpha\,, \tag{2}$$

Where ϕ represents measurements matrix, $y(n) \in R^n$, n=1,2,…M, $M<<N$ represents measurements and product $\phi\psi=\theta$ is reffered as CS matrix. Since $M<<N$ signal reconstruction requires solving undetermined system of equations, which is usually solved by searching for the optimal solution that satisfies the equation:

$$\min_X \|X\|_1 \quad s.t. \quad y=\theta x \tag{3}$$

B. *Compressive sensing based on the demodulation of multicomponent signals*

Achieving the sparsity of multicomponent signal in a certain transform domain is a challenging task, due to mutual interference of different components. In other words, it is difficult to provide the domain in which the multicomponent signals exhibit sparsity so that the CS reconstruction can be applied. It is widely known that in the Fourier domain we can capture sinusoidal features of signal component and those can be generally reconstructed using the standard CS algorithms. In order to provide the sinusoidal behavior for other types of components in DFT domain, we use components demodulation

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approach as in [15]. Let us observe signal x that consists of P components with polynomial and cosine phase modulation:

$$x(n)=\sum_{i=1}^{P}x_i(n)=\sum_{i=1}^{P}A_i e^{j\frac{2\pi}{M}b_i\cos(\frac{2\pi a_i n}{M})+j\frac{2\pi}{M}c_i n} +B_i e^{j\frac{2\pi}{M}(nd_{1i}+\frac{n^2 d_{2i}}{2}+...+\frac{n^m d_{mi}}{m!})}, \quad (4)$$

where coefficients a_i, b_i, c_i, d_i are integers, n is discrete parameter, A_i and B_i represent amplitudes and M is the signal length. The DFT of signal x is given by:

$$X(k)=\sum_{n=0}^{M-1}\sum_{i=1}^{P}x_i(n)=\sum_{i=1}^{P}A_i e^{j\frac{2\pi}{M}b_i\cos(\frac{2\pi a_i n}{M})+j\frac{2\pi}{M}c_i n}e^{-j\frac{2\pi}{M}nk}+ +B_i e^{j\frac{2\pi}{M}(nd_{1i}+\frac{n^2 d_{2i}}{2}+\frac{n^m d_{mi}}{m!})}e^{-j\frac{2\pi}{M}(\frac{n^2 k_2}{2}+...+\frac{n^m k_m}{m!})}e^{-j\frac{2\pi}{M}nk} \quad (5)$$

As it can be seen, signal is not sparse in DFT and CS technique is still not usable. With demodulation we can eliminate unwanted part of signal and achieve sparsity. Because there are two different types of components in signal (component with cosine phase and component with polynomial phase) we have used two different types of demodulation terms:

$$\delta(n)=e^{-j\frac{2\pi}{M}b\cos(\frac{2\pi an}{M})} \quad (6)$$

for component with cosine phase modulation and:

$$\sigma(n)=e^{-j(\frac{d_2 n^2}{2}+...+\frac{d_m n^m}{m!})} \quad (7)$$

for component with polynomial phase modulation. Note that demodulation process is applied "step by step" meaning that the signal will be demodulated on a component by component basis. For instance, parameters (a, b) are changed until they are match with one pair of phase parameters (a_1, b_1),…,(a_P, b_P). The corresponding component will be then demodulated and reduced to sinusoid. It further means that the component on the corresponding frequency will be dominant in the spectrum (the amplitudes of components are usually assumed to be close to each other). The procedure is the same with polynomial phase modulation, i.e., when set (d_2,..., d_P) matches one of the component with parameters (d_{2i},.., d_{mP}) in (5), this component is demodulated and reduced to sinusoid.

Therefore, we can write two demodulations:

$$s=x\delta \quad and \quad z=x\sigma \quad (8)$$

where s is used for component with cosine phase modulation and z for component with polynomial phase modulation. Now, let us assume the CS scenario, meaning that we are dealing with an incomplete set of random samples – measurements. The measurement matrix is denoted as Φ. Assuming that the demodulated component is sparse in DFT domain since it is reduced to the dominant sinusoid (while interferences are much smaller), then we can write:

$$y_s=\Phi s=\Phi F^{-1}S=\Theta S; \quad y_z=\Phi z=\Phi F^{-1}Z=\Theta Z \quad (9)$$

where S and Z are DFT vectors corresponding to s and z. When phase parameters of certain component are matched, then either S or Z can be observed as almost sparse representation of the i-th signal component which is certainly disturbed by other components due to interference. Now the idea is to explore if it is possible to reconstruct separately each sparsified demodulated component, and then to perform the re-modulation toward the original signal based on the detected phase parameters. In that sense, in the next section we consider some of the common CS reconstruction approaches.

III. Compressive Sensing reconstruction Algorithms

A. l_1 minimization - basis pursuit primal dual

CS approach uses optimization algorithms in order to find the sparsest solution among large number of possible solutions. In the cases of noise free signals, l_0 norm minimization can be used for providing the unique solution:

$$\hat{x}=\min_x \| x \|_0 \text{ subject to } y=\theta x. \quad (10)$$

However, this optimization problem is NP hard and does not give satisfactory results in nearly sparse signals. Therefore, in the applications, the l_1- norm minimization is used instead:

$$\hat{x}=\min_x \| x \|_1 \text{ subject to } y=\theta x. \quad (11)$$

By introducing variable t and avoiding absolute value in relation (11), l_1 minimization can be recast as linear program:

$$\min_t \sum t \quad \text{subject to} \quad y=\theta x \wedge x\le t \wedge -x\le t. \quad (12)$$

The problem is reduced to the minimization of the following function:

$$\Xi(x,t,b,p,q)=f(t)+b(\theta x-y)+p(x-t)+q(-x-t), \quad (13)$$

where $x=\theta^T y$, $p=-1/(x-t)$, $q=-1/(-x-t)$ and $b=\theta^T(p-q)$. Variables x, t, b, p and q are iteratively updated. Algorithm stops if maximum number of iterations or sufficient accuracy is obtained.

B. Orthogonal Matching Pursuit (OMP)

The basic idea of this algorithm is an iterative estimation of values of vector x. The OMP estimates the magnitude of the nonzero coefficients of signal in the DFT domain by solving the least square error between the orthogonal projection of the recovered signal and measurement vector y. The procedure can be summarized as follows:

- **First step:** Initializing of the residual $r_0=y$, the initial solution x_0=0 and $\Upsilon_0=\varnothing$. Start counter $i=1$.
- **Second step:** $\Upsilon_i=\Upsilon_{i-1}\cup\arg\max_k\left|\langle r_{i-1},\theta_k\rangle\right|$ and
$$x_i=\arg\min_i\left\|r_{i-1}-\Upsilon_i x_{i-1}\right\|_2^2$$
- **Third step**: Update residual $r_i=r_{i-1}-\Upsilon_i x_{i-1}$.
- **Fourth step:** The algorithm terminates until residual falls below determined threshold. If the stopping criterion is not satisfied, increment counter and return to the second step. Under appropriate stopping conditions OMP will recover all significant signal components [7].

C. Single-Iteration reconstruction algorithm (SIRA)

SIRA is the reconstruction algorithm which is based on the analysis of effects caused by missing samples in the observation domain [14].

- **First step:** Set high value for the probability $\boldsymbol{P_e}$**=10**$^{-2}$.
- **Second step:** Calculate optimal number of available samples using A_{min} - the minimum amplitude of components:

  $M_{opt} \geq \arg\min\{P_e\}$, where $P_e = 1-(1-e^{-MA_{\min}/\sigma^2})^N$.
- **Third step:** Calculate the full DFT vector $\mathbf{X}$ from the set of M_{opt} available measurements: $\mathbf{X}=DFT\{\mathbf{y}\}$.
- **Fourth step:** Determine positions of DFT components higher than $\sqrt{-\sigma^2\log(1-\sqrt[N]{P_e})}$:

  $$\mathbf{k}=\arg\left\{|X|>\sqrt{-\sigma^2\log(1-\sqrt[N]{P_e})}\right\}.$$
- **Fifth step:** Calculate exact DFT values at positions $\mathbf{k}$:

  $$\mathbf{X}=(\Theta^*\Theta)^{-1}\Theta^*\mathbf{y}.$$

where $\boldsymbol{\Theta}$ is obtained as partial DFT matrix (rows correspond to frequencies $\boldsymbol{k}$ and columns to M measurements). Unlike the OMP and l_1 minimization, this algorithm have threshold that detects only the sparse components of signal which means that only dominant component will be reconstructed and not the ones resulted from interference.

IV. PERFORMANCE ANALYSIS OF CS ALGORITHMS APPLIED TO SIGNALS WITH NONLINEAR PHASE MODULATION

Let us observe multicomponent signal:

$$x(t) = 3e^{-j\pi T\cdot 7(2t^2+t)} + 2e^{-j\cos(0.75T\cdot 2\pi\frac{t}{2})+jtT\cdot 15\pi},$$

where t=[-1/2, 1/2] with step Δt=1/1024 and T=32. The length of the signal is M=1024. The original signal is not sparse and needs to be sparsified in order to apply CS method. By demodulating components of interest, the reconstruction problem is reduced to the sinusoid reconstruction. The demodulation is done twice (for component with polynomial and cosine phase), with different demodulation terms. The procedure can be observed through the following steps:

1) Available samples are first multiplied with $e^{(2j\pi pTt^2)}$ in order to perform demodulation of polynomial phase modulated signal part. The parameter search method is done by iteratively changing parameter p in the range [p_{min}, p_{max}].

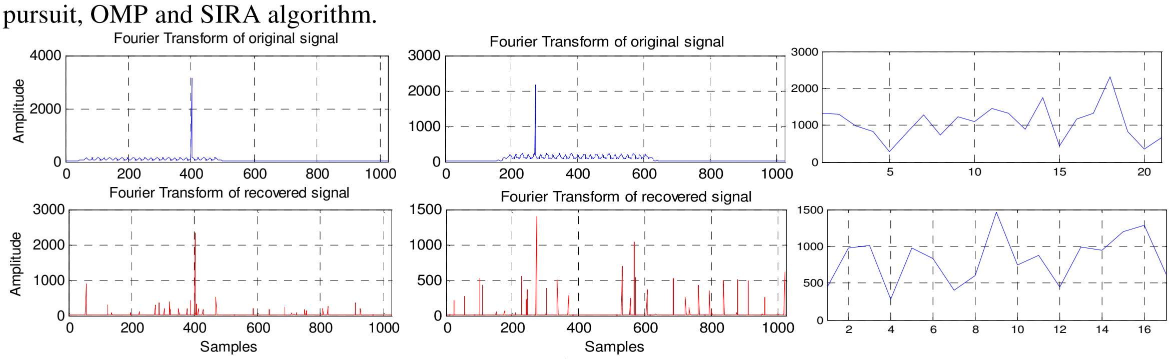
2) Demodulated signal is then reconstructed with three different reconstruction algorithms: l_1-minimization primal dual basis pursuit, OMP and SIRA algorithm.

3) The position of the peak value is stored for each of the considered algorithms, in each iteration.

4) The procedure is repeated for new value of the parameter p, as long as $p<p_{max}$ holds.

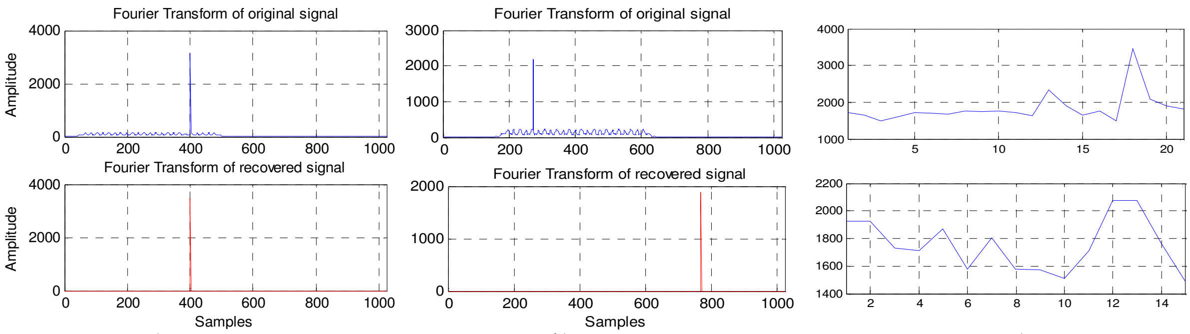
5) After all iterations are finished, the position of the maximum among all peak values (for one type of reconstruction algorithm) should reveal the exact value of parameter p corresponding to the polynomial component.

After demodulation of polynomial phase modulated part of signal, signal is multiplied with $e^{(2j\cos(2\pi qTt/2))}$ in order to perform demodulation of cosine phase modulated part. The true value of the parameter q is searched similarly as in the steps 2)-5), using the values in the range [q_{min}, q_{max}]. We have used set [-10 10] for matching the parameter p, while the value for parameter q was searched from the set [-0.25 6.25]. Values p=7 and q=0.75 are obtained during the parameter search procedure and correspond to the true signal parameters (frequency position 3 corresponds to the p=7, while frequency position 18 corresponds to the q=0.75). During the step 2), reconstruction is done with three mentioned algorithms and different number of samples N (between 32 (3%) of total signal length and 120 (11,71%) of the total signal length). The results are given in Figs. 1-3 for N=32. As it can be seen from the Fig. 1, l_1 minimization based on the primal dual basis pursuit, fails to reconstruct signal due to the large number of nonzero coefficients present in the spectrum (for chirp demodulated signal, nonzero coefficients in the spectrum are consequence of the cosine modulated component, and vice versa). The OMP detects chirp parameter easily, but fails to reveal the exact value of the cosine parameter. However, it has been experimentally shown that the performance of OMP can be improved by increasing the number of available samples. Finally, we may observe that the SIRA algorithm can perfectly reconstruct signals that are approximately sparse and return the exact values for both, p and q parameters at positions 3 and 18, Fig.3. After detecting the polynomial and cosine signal parameters, the re-modulation of the reconstructed sinusoids is performed, using matched parameters. As a result of re-modulation, the reconstructed signal components are obtained.



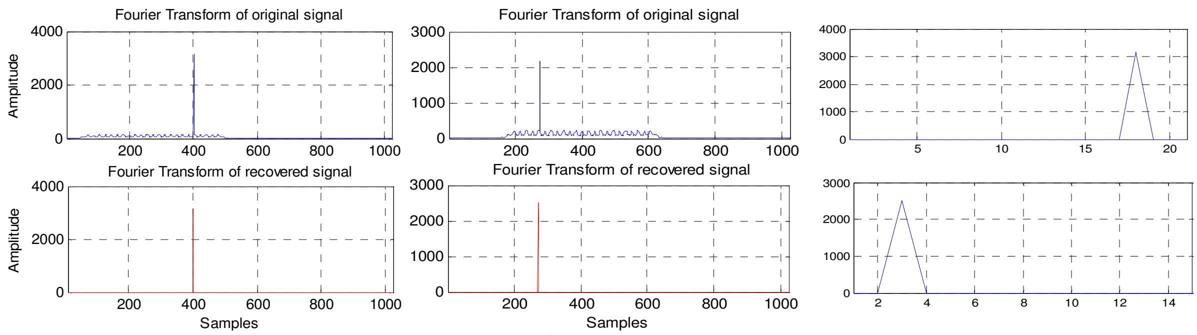
a) b) c)

Figure 1: Reconstruction using l_1-norm minimization approach: a) Chirp component, b) Cosine modulated phase component, c) max amplitudes of reconstruction results for different values of p (upper plot) and q (lower plot)



a) b) c)

Figure 2: Reconstruction using OMP: a) Chirp component, b) Cosine modulated phase component, c) max amplitudes of reconstruction results for different values of p (upper plot) and q (lower plot)



a) b) c)

Figure 3: Reconstruction using SIRA: a) Chirp component, b) Cosine modulated phase component, c) max amplitudes of reconstruction results for different values of p (upper plot) and q (lower plot)

V. CONCLUSION

The method for CS reconstruction of the signals with cosine and polynomial phase modulations is proposed. Considered signals do not satisfy sparsity property and have to be sparsified prior to the reconstruction is done. The sparsification is done through demodulation, by using direct parameter search method, for both, cosine and polynomial phase modulations. After the demodulation, signal is reconstructed with three commonly used reconstruction algorithms. It is shown that, by using small number of samples (3%), only SIRA algorithm succeed in detecting both, polynomial and cosine signal parameters. These parameters are then used in re-modulation process, in order to recover original signal.

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