L-Statistic Combined with Compressive Sensing

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ABSTRACT

Robust signal analysis based on the L-statistic was introduced for signals disturbed with high additive impulse noise. The basic idea is that a certain, usually large number of arbitrary positioned signal samples is declared as heavily corrupted by noise. Then, these samples are removed. Thus, they can be considered as absent or unavailable. Hence, the L-statistics significantly reduces the number of available signal samples. Moreover these samples are randomly distributed, so an efficient analysis of such signals invokes the compressive sensing reconstruction algorithms. Also, it will be shown that the variance of noise, produced by missing samples, can be used as powerful tool for signal reconstruction. Additionally, in order to provide separation of stationary and nonstationary signals the L-statistic is combined with compressive sensing algorithms. The theoretical considerations are verified by various examples, where discrete forms of the Fourier transform and short-time Fourier transform are used to demonstrate the effective integration of the two techniques.

Keywords: L-statistics, compressive sensing, noise modeling, signal separation

1. INTRODUCTION

The L-statistics based approaches are introduced to deal with signals affected by significant amount of impulsive noise[1]-[3]. In order to provide an efficient analysis of stationary and nonstationary signals, numerous versions of the L-estimate transforms are defined. All of them are relied on a trimming procedure that aims to remove corrupted samples[2]. After these samples are removed, an incomplete set of randomly distributed signal measurements is obtained. Thus, the L-statistic produces the same effect as the compressive sampling. Namely, in both cases we deal with a set of measurements which is significantly reduced comparing to the full data set, acquired at the Nyquist-Shannon sampling rate[4]-[9]. Therefore, it is important to ensure that the available number of samples provides an accurate signal estimation and reconstruction. Moreover, the L-estimate algorithms and compressive sensing can be combined to produce a powerful tool for analysis of signal corrupted by a high amount of impulsive noise.

In the recently published paper[10], it has been shown that the omitted samples generate a certain kind of noise with variance that can lead to the false estimation or reconstruction in some cases. In this paper, we will show that instead of commonly used correlation procedures, this variance can be used as a crucial step for the signal reconstruction. Further, the L-statistics combined with a compressive sensing algorithm is applied to the problem of signal separation. Namely, a stationary part of signal is separated from its nonstationary part, including the case when a significant overlapping in the time-frequency domain appears. The theory is illustrated by carefully selected and practically oriented examples, including separation and reconstruction of rigid body from the micro-Doppler components in radar signal[11]-[13]. This example shows that the proposed separation technique is superior over the existing methods, even comparing to the case of using an ideal notch filter.

The paper is organized as follow. After Introduction, an analysis of noise produced by missing samples is considered in Section 2. Then a signal reconstruction method based on the noise variance is derived in Section 3. The signal separation based on the combined L-statistics and compressive sensing is given in Section 4. In Section 5, the presented theory is verified by examples. Concluding remarks are provided in the last section.

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2. ANALYSIS OF NOISE PRODUCED AS A SIDE EFFECT OF MISSING SAMPLES

In the case of compressive sampling, or when trimming signals using the L-estimation, the number of available samples is much fewer than N. For instance, we assume that for a considered signal $x(n)$ with K>1 frequency components, the number of available samples is M, where M>K holds. The samples are chosen randomly in M time points and their values are defined by the set:

$$x(n_1), x(n_2), x(n_3), ..., x\left(n_M\right). \tag{1}$$

In order to analyze the impact of removing samples in the L-estimation process, let consider the standard FT of the non-noisy sinusoidal signal of the form $\exp(j2\pi k_0 m/N)$:

$$X(k)=\sum_{m=0}^{N-1} x(m)e^{-j2\pi mk/N} = \sum_{m=0}^{N-1} e^{j2\pi m(k-k_0)/N} . \tag{2}$$

For an integer k_0, it reduces to delta function. If we remove arbitrary positioned terms, the resulting FT $X_L(k)$ will have random properties. Consider, for example, that N_Q samples have been removed so that N-N_Q terms remain. Then, depending on the value of k, two cases may arise.

Case 1: k=k_0 corresponds to the frequency of signal. At this frequency all terms within the sum are the same and equal to 1. Thus, the value of $X_L(k_0)$ does not depend on the positions of the removed samples and it is given by:

$$X_L(k_0)=(N-N_Q). \tag{3}$$

Case 2: k=l+k_0, l≠0. The removed samples assume values from set $\Theta=\left\{x_l(m)=\exp(j2\pi ml/N),\ m=0,1,...,N-1\right\}$, with equal probability, for a given frequency $l=k-k_0$. The removed samples produce effect that can be modeled as an additive noise: $\varepsilon_L(m)=-x_l(m)$. Statistical mean of these values (with respect to m) is $E\{x_l(m)\}=0$ for l≠0, leading to $E\{X_L(l+k_0)\}=0$. The resulting statistical mean for any k is:

$$E\left\{X_L(k)\right\}=(N-N_Q)\delta(k-k_0). \tag{4}$$

The variance of the random variable taking values from the set Θ is $var\{x_l(m)\}=1$. Taking into account that variables in Θ are not independent (since their complete sum over all N samples is zero), the variance of $X_L(k_0 + l)$ is:

$$\sigma_r^2 = \mathrm{var}\{\sum_{m=1}^{N_Q} x_l(m)\} = N_Q(1-\frac{N_Q-1}{N-1}) \quad for \quad l\neq 0. \tag{5}$$

The DFT behaves as if the signal was noisy with resulting variance σ_r^2. For $N_Q \ll N$ we get $\sigma_r^2 \cong N_Q$. The ratio of $X_L(k)$ at $k \neq k_0$ and $X_L(k_0)$ satisfies:

$$\left|\frac{X_L(k)}{X_L(k_0)}\right| < \frac{\sqrt{6}}{(N-N_Q)}\frac{\sigma_r}{\sqrt{2}} = \sqrt{\frac{3N_Q}{(N-N_Q)(N-1)}}, \tag{6}$$

with probability of 0.95. Here, we assumed that large number of terms is removed. Thus, according to the central limit theorem, the sum behaves as Gaussian random variable (its real and imaginary part, while the absolute value is Rayleigh distributed). As an example, consider N-N_Q=*32* and N=256. Then, $|X_L(k)/X_L(k_0)|$<0.27 with probability 0.95, meaning that 5% values of $X_L(k)$ are above 0.27$X_L(k_0)$ due to this source of error. Having in mind the DFT linearity, this error analysis can be easily generalized for a sum of K sinusoidal signals.

This analysis can be used as a guideline toward the answer on the question: how much samples can be omitted so that we still have an efficient CS reconstruction or L-statistics based estimation? Namely, if a noise level, caused by missing samples, exceeds the level of signal than reconstruction/estimation will be inefficient. Furthermore, the presented analysis can be generalized to provide probability of false reconstruction/estimation in terms of the number of unavailable samples.

In Section 4 (Figs.1 and 2) we prove the efficiency of the proposed combined technique when dealing with signals affected by a high amount of impulsive noise.

3. CS RECONSTRUCTION OF SPARSE SIGNALS BASED ON THE VARIANCE ESTIMATION

The previous analysis can be extended in order to provide an efficient algorithm for sparse signal reconstruction. Namely, we can calculate a generalized deviation in the form:

$$F\{e(n_m,k)\}=F\{|x(n_m)exp^{-j2\pi kn_m/N}-X(k)|\} \quad (7)$$
$$n_m \in N_{avail}=\{n_1,n_2,...,n_M\}$$

where $F\{\ \}$ is a generalized loss function corresponding to a certain norm. Note that if the error based on the ℓ_2 norm is used, the standard DFT is obtained. In that case, we have the estimated DFT in the form:

$$\widehat{X}(k)=\underset{n_m\in N_{avail}}{mean}\{x(n_1)exp^{-j2\pi kn_1/N},...,x(n_M)exp^{-j2\pi kn_M/N}\}.$$

However if we use ℓ_1 norm the robust estimate based on marginal median follows:

$$\widehat{X}(k)=\underset{n_m\in N_{avail}}{median}\{x(n_1)exp^{-j2\pi kn_1/N},...,x(n_M)exp^{-j2\pi kn_M/N}\}.$$

Now we can calculate generalized the error:

$$e(n_m,k)=|x(n_m)exp^{-j2\pi kn_m/N}-\widehat{X}(k)|^l.$$

If the norm ℓ_2 is used, this error will be the variance. After it is calculated for each available sample, we can obtain the variance of error for each frequency k=0,1,...,N, as:

$$V(k)=\text{var}\{e(n_1,k),e(n_2,k),...,e(n_M,k)\}, \quad (8)$$
$$\text{for } k=1,2,...,N.$$

Depending whether the frequencies k=1,2…,N belong to the signal components $k=k_1$ or not, the variance vector $v(k)$ can be estimated as follows:

$$\text{var}\left\{e(n_1,k_{0p}),e(n_2,k_{0p}),...,e(n_M,k_{0p})\right\}=N_A\frac{N-N_A}{N-1}A_2^2+...+N_A\frac{N-N_A}{N-1}A_K^2 \quad (9)$$

$$\text{var}\left\{e(n_1,k_{0q}),e(n_2,k_{0q}),...,e(n_M,k_{0q})\right\}=N_A\frac{N-N_A}{N-1}A_1^2+...+N_A\frac{N-N_A}{N-1}A_K^2, \quad (10)$$

where A_i denotes the amplitude of the i-th signal component, and the K sparse signal is of the form:

$$x(n)=\sum_{i=1}^{K}A_i exp(j2\pi k_{0i}n/N+\varphi_i).$$

Therefore, the frequencies belonging to the signal components can be simply obtained as a solution of the following minimization problem:

$$k_{0i}=\arg\min\{V(k)<T\},\ for\ k=1,...,N, \quad (11)$$

where T represents a certain threshold which can be calculated with respect to the max{$V(k)$}, e.g. αmax{$V(k)$} (α is a constant between 0.85 and 0.95). The CS matrix is formed starting from the DFT matrix, from which we keep only the columns corresponding to the available measurements $n_m \in N_{avail}=\{n_1,n_2,...,n_M\}$ and the rows that corresponds to the extracted frequencies k_{0i}. Since there are more equations than unknowns, the system $AX=B,\quad or\ XA^T=B^T$, is solved in the least square sense, by using MATLAB operation or by using pseudo inversion:

$$X=B^T/A^T \quad or \quad X=(A^*A)^{-1}A^*B. \quad (12)$$

4. SIGNAL SEPARATION BY COMBINING L-ESTIMATION AND COMPRESSIVE SENSING

Consider a signal $x(n)$ that consists of two parts: stationary and nonstationary. Let assume that a significant overlapping in the time-frequency domain exist. The goal is to separate and reconstruct the stationary part of signal. Note that this problem formulation is inspired by real problem in radar where the rigid body component should be extracted from the micro-Doppler components. Observe that the time-frequency based filtering, combined with high resolution techniques[14]-[16], cannot provide satisfactory results because the signal will be significantly changed on the points where signals overlap. The compressive sensing techniques cannot be directly applied, because the considered signal is not sparse either in time or frequency domain. However, when the stationary part is observed separately, it is sparse in the frequency domain. Thus, we should provide a separation of these two parts in the time-frequency domain, even at the cost of losing certain portions of desired stationary components. For that purpose we will use the short-time Fourier transform (STFT) and the L-estimation procedure. Namely, the STFT provides a cross-terms free time-frequency representation, keeping the phase information. After the L-estimate approach is applied, only the STFT of stationary signal part will be obtained, but with significant amount of missing coefficients. Observe that these coefficients are randomly distributed over the whole time-frequency plane. Hence, recovering the desired stationary part will be possible by applying compressive sensing reconstruction. For that purpose, we need to establish a relationship between the time-frequency domain as the domain of observation and frequency domain as a domain where this signal is surely sparse.

Therefore, the classical CS problem formulation should be extended for the two-dimensional time-frequency analysis based on the STFT, which is defined as:

$$STFT(n,k)=\sum_{m=0}^{M-1} x(n+m)e^{-j2\pi mk/M}\ , \tag{13}$$

where M is the window width (the rectangular window is assumed). In the matrix form the previous relation can be written as follows:

$$\mathbf{STFT}_M(n)=\mathbf{W}_M\,\mathbf{x}(n), \tag{14}$$

where:

$$\mathbf{STFT}_M(n)=[STFT(n,0),...,STFT(n,M-1)]^T, \tag{15}$$

$$\mathbf{x}(n)=[x(n),x(n+1),...,x(n+M-1)]^T,$$

while W_M is the $M\times M$ DFT matrix with coefficients: $W(m,k)=\exp(-j2\pi km/N)$. Without loss of generality, in the sequel we consider the non-overlapping case of STFT. In that case, at a certain instant $n+M$, the STFT vector will be: $\mathbf{STFT}_M(n+M)=\mathbf{W}_M\,\mathbf{x}(n+M)$. Now, we can combine all STFT vectors in a single equation, as follows:

$$\mathbf{STFT}=\mathbf{W}_{M,N}\,\mathbf{x}, \tag{16}$$

where **STFT** and $\mathbf{W_{M,N}}$ matrix are defined as follows:

$$\mathbf{STFT}=\begin{bmatrix}\mathbf{STFT}_M(0)\\ \mathbf{STFT}_M(M)\\ \dots\\ \mathbf{STFT}_M(N-M)\end{bmatrix},\quad \mathbf{W}_{M,N}=\begin{bmatrix}\mathbf{W}_M & \mathbf{0}_M & \dots & \mathbf{0}_M\\ \mathbf{0}_M & \mathbf{W}_M & \dots & \mathbf{0}_M\\ \dots & \dots & \mathbf{W}_M & \mathbf{0}_M\\ \mathbf{0}_M & \mathbf{0}_M & \dots & \mathbf{W}_M\end{bmatrix}. \tag{17}$$

The vector **x** is given by:

$$\mathbf{x}=[\mathbf{x}(n)^T,\mathbf{x}(n+1)^T,...,\mathbf{x}(n+M-1)^T]^T=$$
$$=[x(0),x(1),...,x(N-1)]^T$$

Furthermore, vector **x** can be expressed using the corresponding DFT vector $\mathbf{X}=[X(0), X(1), \dots, X(N-1)]^T$:

$$\mathbf{x}=\mathbf{W}_N^{-1}\mathbf{X}\ , \tag{18}$$

where $\mathbf{W}_N^{-1}$ denotes the inverse DFT matrix of the dimension $N\times N$. Finally, we can define the relationship between the STFT domain and DFT domain in the form:

$$\mathbf{STFT}=\mathbf{W}_{M,N}\,,\mathbf{W}_N^{-1}\mathbf{X}=\mathbf{AX} \quad (19)$$

According to the problem formulation, we aim to separate stationary and sparse components from the nonstationary and nonsparse ones. For that purpose we need to apply firstly the L-estimation approach. The data are sorted in TF domain along time axis for each frequency separately, and then certain amount of highest samples is discarded. Here, we are facing with three specific cases: a) If there are only nonstationary components at a specific frequency, they will be eliminated within the trimmed highest values; b) If both the nonstationary interference and desired sparse components contribute at certain frequency, the highest values would correspond to the overlapping regions as well as to the non-stationary interference alone, and will be removed together; c) Another possible case is when the non-stationary and sparse signals are of the same order of amplitude, but the opposite phases produce low values at the intersection points. In this case, the solution is to remove some of the lowest values, in addition to the highest ones. In all cases, after the L-estimation procedure, we are left only with the stationary components, but with the significantly reduced set of corresponding samples.

The L-statistics approach applied to the **STFT** matrix, separately for each frequency point k_i can be defined by:

$$\mathbf{S}=sort\{\mathbf{STFT}(n,k_i),\ n=0,...,N\text{-}1\}, \quad (20)$$

$$\mathbf{S}_\alpha=\left\{S(p(i)),\ for\ i\in[1,N-N_Q]\right\}, \quad (21)$$

where $\mathbf{p}=\arg\left\{sort\left(\mathbf{STFT}(n,k_i),\ n=0,...,N\text{-}1\right)\right\}$.

A certain number N_Q of the highest elements are removed from consideration, and the resulting matrix of available STFT values is denoted by $\mathbf{S}_\alpha$. The corresponding matrix $\mathbf{A}_\alpha$, relating the sparse DFT vector **X** to $\mathbf{S}_\alpha$, is formed by omitting the rows corresponding to the removed STFT values. Each row corresponds to one time and frequency point (n,k). In order to reconstruct the original sparse signal, which produces the best concentrated $X(k)$, we may define the following minimization problem:

$$\min\|\mathbf{X}\|_{l_1}\quad subject\ to\quad \mathbf{S}_\alpha=\mathbf{A}_\alpha\mathbf{X}\,. \quad (22)$$

Efficiency of the proposed procedure is demonstrated and discussed in the next section, Fig.3 and 4.

5. SIMULATION RESULTS

Example 1: Consider a discrete noisy signal that is consisted of four sinusoidal components:

$$x(n)=e^{j2\pi k_{01}n/M}+e^{j2\pi k_{02}n/M}+e^{j2\pi k_{03}n/M}+e^{j2\pi k_{04}n/M}+\nu(n)$$

where $M=64$, $k_{01}=4$, $k_{02}=16$, $k_{03}=32$, $k_{02}=54$ and $\nu(n)$ is impulse noise that significantly affects up to 35% of data samples (22 samples). Using the L-statistics, 24 values are omitted out of 64 (32% of the data are omitted, while 25% of data are seriously corrupted by noisy pulses). As will be shown, up to about 70% of omitted values will not significantly change the signal reconstruction performance. The STFT is illustrated for a single time instant (FT of one windowed signal part). The standard STFT of non-noisy and noisy signal are shown in Fig. 1a and b, respectively. The L-estimate approach is given in Fig. 1c. The proposed approach is given in Fig. 1d. We can observe that the proposed approach provides almost the same performance as the original signal transform.

The L-estimate STFT and the CS based L-estimate STFT, calculated for the entire signal in impulse noise are shown in Figs. 3a and b, respectively.

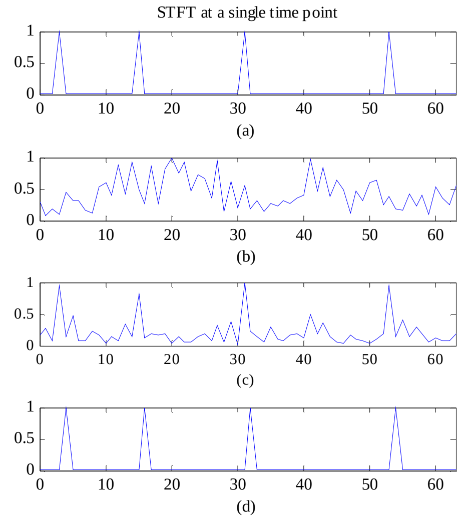

Fig. 1. a) STFT of non-noisy signal, b) STFT of noisy signal, c) L-estimate STFT, d) CS L-estimate STFT

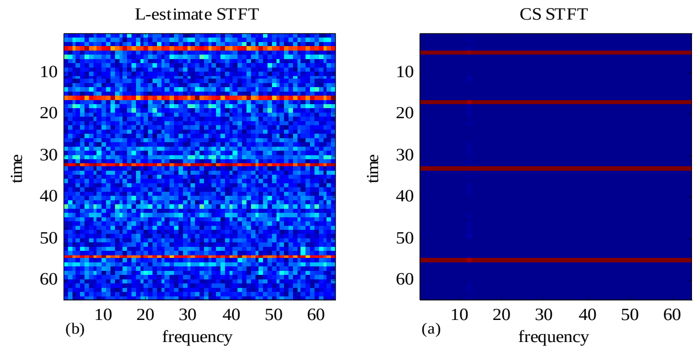

Fig. 2. a) standard L-estimate STFT, b) CS based L-estimate STFT

Example2: This example aims to show the efficiency of the combined L-estimation and compressive sensing approach in signal components separation. The desired signal consists of eight stationary sinusoids, while the non-stationary disturbance is consisted of five sinusoidally modulated signals as well as several short duration pulses and strong transient signals (some of them are at the same frequencies as the stationary sinusoids). The STFT is calculated for N = 1024 and M = 32. The spectrogram of the data is presented in Fig. 3a.

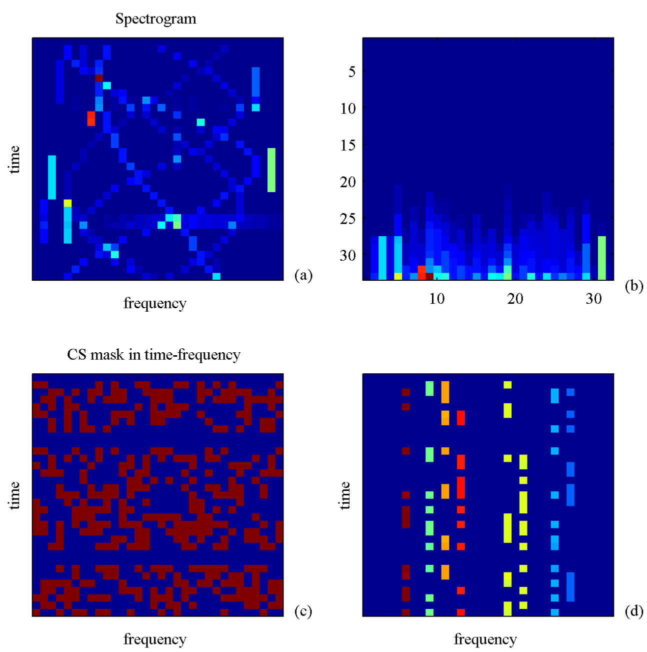

Fig. 3. a) STFT of the composite signal, b) Its sorted values, c) CS mask corresponding to the L-statistics based STFT values, d) STFT values that remain after applying the L-statistics on the absolute values of the STFT (bottom)

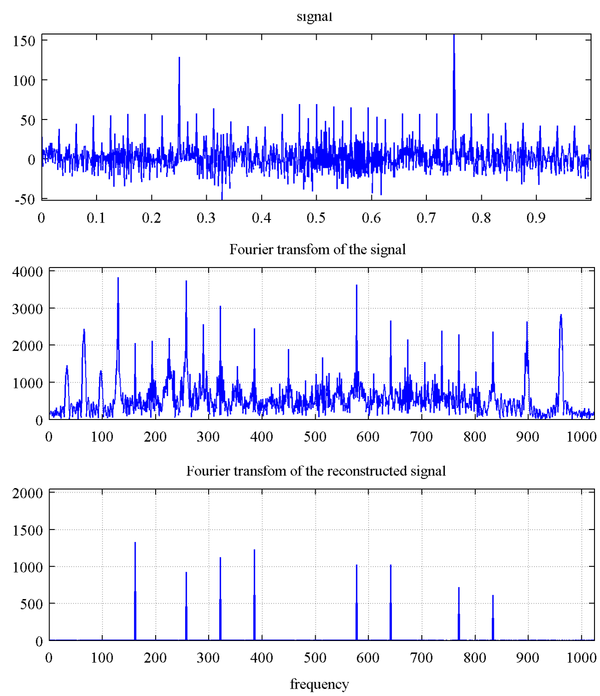

Fig. 4. Signal composed of a sparse part and non-stationary part, with additional transient impulses in time, including two strong pulses (top), Fourier transform of the original composite signal (middle). The reconstructed Fourier transform by using the CS values of the STFT, corresponding to sparse part of the composite signal (bottom)

The L-statistics is performed on the sorted STFT values (Fig. 3b) and 50% of the largest values are removed along with 10% of the smallest values. The CS mask corresponding to the L-statistics based STFT values is shown in Fig.3.c. The CS spectrogram values (that remain after the L-statistics) are shown in Fig. 3d.
The original time domain data are shown in Fig.4 (top). The signal reconstruction is performed based on the STFT values from Fig. 3d. Compared to the DFT of the data in Fig.4 (middle), the reconstructed DFT shown in Fig. 4 (bottom) is equal to the original DFT, preserving amplitude and phase.

6. CONCLUSION

This paper analyses and combines two attractive approaches for robust signal representation and reconstruction. The L-estimation and compressive sensing are applied for signals that are sparse in the Fourier domain. First, we show that the L-estimation can be observed as a specific kind of compressive sampling approach. Second, we derive the statistics of spectral noise that appear as a consequence of missing samples, showing that the noise variance depends on the number of missing sample. Moreover, this variance can be used as a tool for defining the signal reconstruction procedure. Finally, many complex problems in signal processing can benefit from combining L-estimation and compressive sensing, as it is done in the case of signal separation elaborated in this paper.

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