Time-Frequency Rate Distributions with Complex-lag Argument

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Abstract— **A general form of the Nth order complex-lag time-frequency rate distribution is proposed. A few interesting special cases are considered and analyzed. The proposed approach can arbitrarily reduce the spread factor. Hence, it provides a high concentration even for signals with fast varying instantaneous frequency rate.**

I. INTRODUCTION

The instantaneous frequency rate (IFR) is defined as the second derivative of a signal phase. It can be useful in different practical applications such as radar, communications and video surveillance [1], [2]. One of the commonly used time-frequency rate representations is the O'Shea distribution [3]. Also, an IFR estimator for high-order polynomial phase signals has been proposed in [4], [5].
In this Letter we propose a general form of complex-lag time-frequency rate distribution for the IFR estimation. The concept of the complex-lag argument has been introduced to provide highly concentrated time-frequency distributions [6]-[8]. To provide an arbitrary high concentration along the IFR, this concept is extended here to the time-frequency rate representations. The special cases of the proposed general form are considered, as well. It has been shown that, owing to the reduced spread factor, they can be efficiently used for the IFR estimation of frequency modulated signals with fast varying IFR.

II. Theory

The derivatives of a holomorphic function f, defined on the closed disc, are obtained by using the integration over boundary circle C of the disc as follows:

$$f^K(t) = \frac{K!}{2\pi j}\int_C \frac{f(z)}{(z-t)^{K+1}}dz. \qquad (1)$$

Furthermore, we assume that C is centered at the instant t and $z = t + \tau e^{j\theta}$, where τ is the radius of circle and $\theta \in [0, 2\pi]$. When $f(t)=\varphi(t)$ and $K=2$, the IFR is obtained as [7]:

$$\Omega(t) = \varphi^{(2)}(t) = \frac{2!}{2\pi\tau^2}\int_0^{2\pi} \varphi(t+\tau e^{j\theta})e^{-2j\theta}d\theta \qquad (2)$$

where $\varphi(t)$ is a phase of a signal $x(t) = r\cdot e^{j\varphi(t)}$. The discretization o f (2) leads to the following form:

$$\frac{\varphi^{(2)}(t)\tau^2}{2!} = \sum_{k=0}^{N-1} \varphi(t+\frac{\tau}{N}e^{j2\pi k/N})e^{2(-j2\pi k/N)} \qquad (3)$$

Having in mind (3), the Nth order complex-lag signal moment can be defined by:

$$R_x(t,\sqrt{\tau}) = \\ = \prod_{k=0}^{N-1}\left(x\left(t+\sqrt{\frac{2!}{N}\tau e^{j2\pi k/N}}\right)\right)^{e^{2(-j2\pi k/N)}} \qquad (4)$$

Thus, the time-frequency rate distribution is:

$$TFR(t,\Omega) = \int_0^{\infty} R_x(t,\sqrt{\tau})e^{-j\Omega\tau}d\tau$$

$$= \int_0^{\infty} R_x(t,\tau)e^{-j\Omega\tau^2}d\tau. \qquad (5)$$

III. Generalized form of time-frequency rate distributions

The IFR estimation accuracy depends on the number and the choice of points used in discretization of (2). Thus, let us start with the simplest case based on two arbitrary points symmetrical around t: $\tau/(a+jb)$ and$\tau/(-a-jb)$. The corresponding form of the second order time-frequency rate distribution can be defined as:

$$TFR^{N=2}(t,\Omega) =$$
$$= \int_0^{\infty} \left(x\left(t+\frac{\tau}{(a+jb)}\right) \times x\left(t+\frac{\tau}{(-a-jb)}\right)\right)^{(a+jb)^2} e^{-j\Omega\tau^2} d\tau. \quad (6)$$

Owing to the symmetry of points, the phase of the complex-lag signal moment does not contain odd phase derivatives:

$$\varphi_R(t,\tau) = 2(a+jb)^2\varphi(t) + +\varphi^{(2)}(t)\tau^2 + 2\varphi^{(4)}(t)\frac{\tau^4}{4!(a+jb)^2} + ... \quad (7)$$

The spread factor (terms with derivatives different from $\varphi^{(2)}(t)$) can be reduced by appropriate selection of parameters a and b. Furthermore, the concentration can be arbitrarily improved by increasing the number of discretization points, i.e., the distribution order. Therefore, by using $N/2$ pairs of symmetrical points $\tau/(a_i+jb_i)$ and$\tau/(-a_i-jb_i)$, a general form of the Nth order time-frequency rate distribution can be defined as:

$$TFR^{N}(t,\Omega) =$$
$$= \int_0^{\infty} \prod_{i=1}^{N/2} \left(x\left(t+\sqrt{\frac{2}{N}}\frac{\tau}{(a_i+jb_i)}\right) \times x\left(t-\sqrt{\frac{2}{N}}\frac{\tau}{(a_i+jb_i)}\right)\right)^{(a_i+jb_i)^2} e^{-j\Omega\tau^2} d\tau. \quad (8)$$

The signal terms with complex-lag argument can be generally calculated as:

$$x(t \pm (u+jv)\tau) =$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\omega)e^{\mp\omega v\tau}e^{j\omega(t\pm u\tau)}d\omega \quad (9)$$

Some special cases of (8), that could be interesting for practical applications, are analyzed in the following.
Case 1: For N=2 and (a,b)=(1,0), the O'Shea distribution is obtained:

$$TFR^{N=2}(t,\Omega) = \int_0^{\infty} x(t+\tau)x(t-\tau)e^{-j\Omega t^2}d\tau. \quad (10)$$

The spread factor for the O'Shea distribution is obtained as:

$$Q(t,\tau) = 2\frac{\varphi^{(4)}(t)\tau^4}{4!} + 2\frac{\varphi^{(6)}(t)\tau^6}{6!} + 2\frac{\varphi^{(8)}(t)\tau^8}{8!}... \quad (11)$$

Thus, regarding the time-frequency rate concentration, this distribution form can be efficiently used for signals whose phase has derivatives up to the fourth order. *Case* 2: The concentration along the IFR for signals with fast IFR variations can be significantly improved by increasing the distribution order. Thus, for N=4 and (a_1,b_1,a_2,b_2)=(1,0,0,1), the fourth order time-frequency rate distribution is defined [7]. The spread factor is significantly reduced in comparison with the previous case, since it contains only the phase derivatives of order $4n$+2, n = 1,2,.... However, it is interesting that the concentration improvement can be achieved even without increasing the distribution order. Hence, we may observe the following case: N=2 and (a,b)=(0,$1/j\sqrt{j}$). The corresponding signal moment is:

$$R_x(t,\tau) = Ae^{j\varphi_R(t,\tau)} = x^j(t+\sqrt{j}\tau)x^j(t-\sqrt{j}\tau) \quad (12)$$

The moment phase function contains the following derivatives:

$$\varphi_R(t,\tau) = j2\varphi(t) + \varphi^{(2)}(t)\tau^2 - j2\frac{\varphi^{(4)}(t)\tau^4}{4!}... \quad (13)$$

To focus on the second order derivative and to eliminate the influence of some higher order derivatives such as $\varphi^{(4)}(\tau)$, the following modification is introduced:

$$R_x^a(t,\tau) = e^{angle(x^j(t+\sqrt{j}\tau)x^j(t-\sqrt{j}\tau))}. \quad (14)$$

Therefore, the highly concentrated second order time-frequency rate representation is ob-

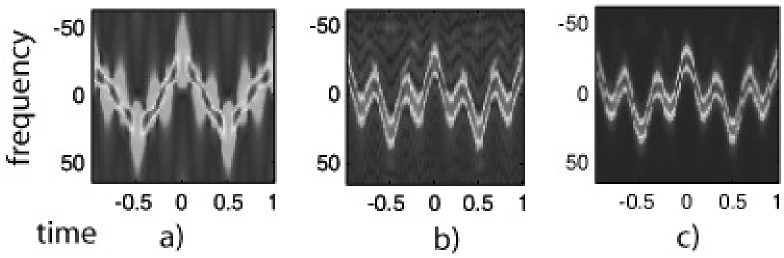

Fig. 1. Time-frequency rate representations: a) $TFR^{N=2}$, b) $TFR_a^{N=2}$, c) $TFR_{a,L=2}^{N=2}$

TABLE I

MSEs of IFR estimation precision

TFR	$TFR^{N=2}$(CP)	$TFR_a^{N=2}$	$TFR_{a,L=2}^{N=2}$
MSE	390	10	4.4

tained as:

$$TFR_a^{N=2}(t,\Omega) = \int_0^\infty R_x^a(t,\tau)e^{-j\Omega\tau^2}d\tau. \quad (15)$$

The spread factor contains only the derivatives of order $4n+2$, as in the case of the fourth order distribution (N=4):

$$Q(t,\tau) = -2\frac{\varphi^{(6)}(t)\tau^6}{6!}$$

$$+2\frac{\varphi^{(10)}(t)\tau^{10}}{10!} - 2\frac{\varphi^{(14)}(t)\tau^{14}}{14!}... \quad (16)$$

Further concentration improvement can be achieved by using the higher order distributions ($N\geq6$) or by using L-form of the distribution (15).

Case 3: The L-form of time-frequency rate distribution can be defined as:

$$TFR_L^N(t,\Omega) =$$

$$= \int_{-\infty}^{\infty} P(\eta)(TFR_{L/2}^N(t,\Omega+\eta)$$

$$\times TFR_{L/2}^N(t,\Omega-\eta))d\eta. \quad (17)$$

where TFR_1^N corresponds to the basic distribution form.

IV. Numerical Example

A periodically frequency modulated signal with fast varying phase function is considered:

$$s(t) = e^{j(4\cos(2\pi t)+1/3\cos(3\pi t)+2/3\cos(6\pi t))}.$$

The signal is calculated for t=-1:Δt:1-Δt, where Δt=2/N, while N=128. The distributions: $TFR^{N=2}$ (Case 1), $TFR_a^{N=2}$ (Case 2), $TFR_{a,L=2}^{N=2}$ (Case 3, the L-form of $TFR_a^{N=2}$) are illustrated in Figs. 1.a-c, respectively. Note that for the considered signal, the O'Shea distribution cannot properly follow the IFR variations and does not provide satisfactory results. Significant improvement is achieved by using the same order distribution $TFR_a^{N=2}$. Further improvement is achieved by using its L-form $TFR_{a,L=2}^{N=2}$. Mean square errors (MSE) of the IFR estimation are given in Table 1.

V. Conclusion

The general form of complex-lag time-frequency rate distribution is proposed. As a special case, the second order optimal distribution is presented. Future research may include similar optimization for higher orders.

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