Effects of Cauchy integral formula discretization on the precision of IF estimation; unified approach to complex-lag distribution and its counterpart L-form

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Abstract— **Effects of Cauchy integral formula discretization on the concentration of time-frequency (TF) distribution are analyzed. As a result of this discretization, new forms of distributions are produced. In order to increase the accuracy of instantaneous frequency (IF) estimation, two solutions are considered: increasing the number of integration points and multiple successive integrations using the same number of points (it corresponds to the L-form of the TF distribution). In practical applications, the L-form of the fourth order complex-lag distribution produces very efficient representation. In this case, the analysis of noise influence is also provided.**

I. INTRODUCTION

The distribution concentration is very important for an efficient TF signal representation [1]-[3]. This is especially emphasized for the IF estimation of highly non-stationary signals [4]-[9]. In order to obtain high concentration in the TF plane, various TF distributions have been proposed [1]-[10].

In this letter we present an approach that can produce highly concentrated representation along a fast varying IF, even when its variations are significant within only a few samples. It is based on the concept of Cauchy's integral formula discretization. In this context, some specific TF distributions of different order are proposed and analyzed. It will be shown that efficiency of the IF estimation can be improved either by increasing the number of discretization points on the circle or by performing multiple successive integrations over the circle. The first solution leads to a definition of novel general forms of TF distributions, whose special cases correspond to the various existing TF distributions [7]-[9]. The second one introduces their L-forms. Here, it is important that multiple successive integrations approach does not significantly influence the realization complexity, since it can be obtained recursively from the basic distribution form as in [7].

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II. Interpreting and Improving TF Distrubution Concentration

A. IF Calculation based on the Cauchy's integral formula

According to the Cauchy's integral formula from the complex analysis, a holomorphic function $f(z)$ defined on the closed disc $D = \{z : |z - z_0| \leq r\}$ can be completely determined by its values on the boundary circle C of the disc:

$$f(t) = \frac{1}{2\pi j} \oint_C \frac{f(z)}{z-t} dz. \tag{1}$$

Consequently, derivatives of the function f are obtained using the integration over C as follows:

$$f^K(t) = \frac{K!}{2\pi j} \oint_C \frac{f(z)}{(z-t)^{K+1}} dz. \tag{2}$$

Let us consider a signal in the form $s(t) = r \cdot e^{j\phi(t)}$, with a constant amplitude r and a phase function $\phi(t)$. In order to provide an estimation of the first phase derivative$\phi^{'}(t)$ i.e. IF, we may assume that circle C is centered at the instant t and $z = t + \tau e^{j\theta}$, where τ is the radius of the circle, while θ is the angle ($\theta \in [0, 2\pi]$) [9]. According to (2), when $f(t)$=$\phi(t)$ and K=1, the IF can be obtained as:

$$\phi^{'}(t) = \frac{1}{2\pi j}\oint_C \frac{\phi(t+\tau e^{j\theta})}{(\tau e^{j\theta})^2} d(\tau e^{j\theta})$$

$$= \frac{1}{2\pi\tau}\int_0^{2\pi} \phi(t+\tau e^{j\theta})e^{-j\theta}d\theta. \quad (3)$$

Reducing the circle radius τ by a scaling factor L results in multiple (L) successive integrations as follows:

$$\phi^{'}(t)\tau = \frac{L}{2\pi}\int_0^{2\pi} \phi(t+\frac{\tau}{L}e^{j\theta})e^{-j\theta}d\theta. \quad (4)$$

In the TF analysis, τ has been introduced into the definition of the TF distributions as a lag coordinate. Thus, scaling of the lag coordinate by L provides increasing of accuracy of the IF estimation. The discretization of (3) leads to the form [9]:

$$\phi^{'}(t)\tau = \sum_{k=0}^{N-1} \phi(t+\frac{\tau}{N}e^{j2\pi k/N})e^{-j2\pi k/N}. \quad (5)$$

For large N, almost an ideally concentrated IF could be obtained. Note that the right side of (5) represents the phase of the complex-lag signal's moment [9]:

$$M(t,\tau) = \prod_{k=0}^{N-1} s^{e^{-j2\pi k/N}}(t+\frac{\tau}{N}e^{j2\pi k/N}). \quad (6)$$

The corresponding TF distributions are obtained as the Fourier transform of the moment function.

B. Analysis of IF estimation precision

In this section an analysis of the IF estimation accuracy, depending on the number and the choice of discretization points, is provided.

Consider the general case of two points on the circle: $\tau/(2(a+jb))$ and $\tau/(2(-a-jb))$, located symmetrically around an instant t, Fig 1.a. In this case the moment function can be written as:

$$M(t,\tau) = s^{(a+jb)}\left(t+\frac{\tau}{2(a+jb)}\right)$$
$$\times s^{(-a-jb)}\left(t+\frac{\tau}{2(-a-jb)}\right). \quad (7)$$

The Taylor series expansion for the phase of $M(t,\tau)$ will produce:

$$\phi_M(t,\tau) = \phi^{'}(t)\tau + \phi^{(3)}(t)\frac{\tau^3}{2^2 3!(a+jb)^2}$$
$$+\phi^{(5)}(t)\frac{\tau^5}{2^4 5!(a+jb)^4} + ... \quad (8)$$

Observe that all the terms except $\phi^{'}(t)\cdot\tau$ (terms containing the third $\phi^{(3)}(t)$, the fifth $\phi^{(5)}(t)$, and higher order odd phase derivatives) represent the integration error. This error is reflected as a spread factor that affects the distribution concentration. The TF representation based on the moment function (7) is defined as:

$$GCD_{N=2}(t,\omega)\,|_{a,b} = \int_{-\infty}^{\infty} s^{(a+jb)}\left(t+\frac{\tau}{2(a+jb)}\right)$$
$$\times s^{(-a-jb)}\left(t+\frac{\tau}{2(-a-jb)}\right)e^{-j\omega\tau}d\tau. \quad (9)$$

Obviously, it provides an ideal concentration for the linear IF. In the sequel, we will present a few interesting cases.

Case 1: Taking (a,b)=(1,0), the real valued points $\tau/2$ and $-\tau/2$ are considered (Fig 1.b). In this case, (9) corresponds to the well-known Wigner distribution. Note that the power -1 is used instead of a conjugate and it has the same influence on the IF estimation, although the marginal properties are not preserved. The spread factor for the Wigner distribution is given in Table I.

Case 2: Regarding the IF estimation, almost the same results as in the previous case are obtained for (a,b)=(0,1) (the points $\pm j\tau/2$ are used, Fig 1.c). The corresponding signal moment is: $M(t,\tau) = s^{-j}(t+j\tau/2)s^{j}(t-$

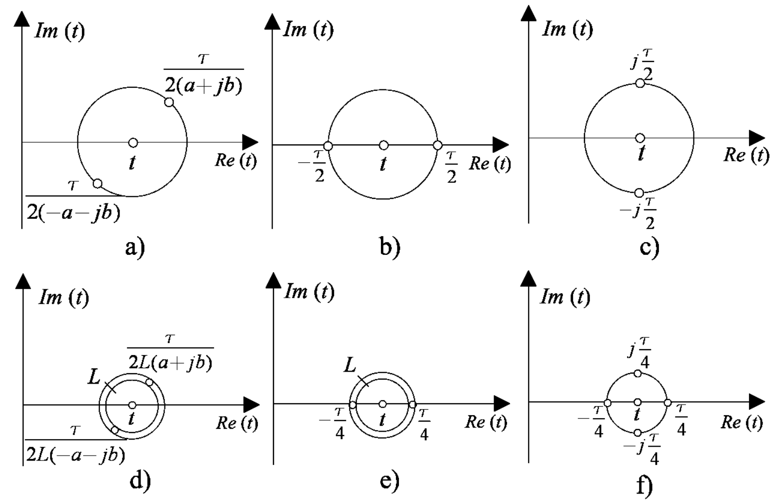

Fig. 1. Illustration of circle points around instant t : a) two points- general case, b) points on real axes, c) points on imaginary axes, d) two points L-times, e) two points on real axes L-times, f) four points

TABLE I
Spread factors for some TF distributions

Distribution	**Spread factor**
Wigner distribution,(GCD_N, N=2)	$Q(t,\tau)=\phi^{(3)}(t)\frac{\tau^3}{2^2 3!}+\phi^{(5)}(t)\frac{\tau^5}{2^4 5!}+...$
L-Wigner distribution $GCD^L_{N=2}$	$Q(t,\tau)=\phi^{(3)}(t)\frac{\tau^3}{2^2 3! L^2}+\phi^{(5)}(t)\frac{\tau^5}{2^4 5! L^4}+...$
$GCD_{N=4}$	$Q(t,\tau)=\phi^{(5)}(t)\frac{\tau^5}{4^4 5!}+\phi^{(9)}(t)\frac{\tau^9}{4^8 9!}+...$
$GCD^{L=2}_{N=4}$	$Q(t,\tau)=\phi^{(5)}(t)\frac{\tau^5}{4^6 5!}+\phi^{(9)}(t)\frac{\tau^9}{4^{12} 9!}+...$
$GCD^L_{N=s}$	$Q(t,\tau)=\phi^{s+1}(t)\frac{\tau^{s+1}}{s^s (s+1)! L^s}+\phi^{2s+1}(t)\frac{\tau^{2s+1}}{s^{2s}(2s+1)! L^{2s}}+...$

$j\tau/2$). Its phase contains the same derivatives as in the Wigner distribution.

Case 3: An interesting case that eliminates the 3^{rd}, 7^{th}, 11^{th} and higher order derivatives can be obtained for a=b=$\sqrt{2}/2$. The Taylor series expansion of the phase function is: $\phi_M(t,\tau)=\phi'(t)\tau-j\phi^{(3)}(t)\frac{\tau^3}{2^2 3!}-\phi^{(5)}(t)\frac{\tau^5}{2^4 5!}+ ....$ Apparently, the derivatives of order $4n$-1 (n=1,2,...) produce amplitude modulation terms. Thus, to avoid these terms for the IF estimation, a slight modification of $M(t,\tau)$ should be used:$e^{j\cdot angle(M(t,\tau))}$.

The spread factor might be significant for a non-linear IF law and the inner interferences appear. Therefore, in order to improve concentration for non-linear phase, a more accurate integration should be considered. It can be realized as the integration over L circles with L times smaller radius, as it is given by (4). An illustration for two arbitrary points is shown in Fig 1.d. The L-form of the distribution, defined by (9), is:

$$GCD^L_{N=2}(t,\omega)\,|_{a,b} = \\ = \int_{-\infty}^{\infty} s^{L(a+jb)}\left(t+\frac{\tau}{2L(a+jb)}\right) \\ \times s^{L(-a-jb)}\left(t+\frac{\tau}{2L(-a-jb)}\right)e^{-j\omega\tau}d\tau. \quad (10)$$

For instance, in the case L=2, the precision of IF estimation is improved by using double integration over the points on the twice smaller radius. Hence, for the points $\tau/4$ and $-\tau/4$ (Fig 1.e), the L-Wigner distribution is obtained as [7]:

$$GCD_{N=2}^{L=2}(t,\omega)=$$

$$=\int_{-\infty}^{\infty} s^2(t+\frac{\tau}{4})s^{-2}(t-\frac{\tau}{4})e^{-j\omega\tau}d\tau. \quad (11)$$

The L-Wigner distribution reduces the spread factor better than the Wigner distribution (Table I), providing higher distribution concentration. The precision of integration (i.e. IF estimation) can be further improved by increasing the number of points, instead of performing multiple successive integrations within only two points. For two pairs of symmetrical points $[\pm\tau/(4(a_i+jb_i))]$, $i=1,2$, the moment's phase expansion is given by:

$$\phi_M(t,\tau)=\phi^{'}(t)\tau+$$

$$+2\phi^{(3)}(t)\frac{\tau^3}{4^3 3!}\left(\frac{1}{(a_1+jb_1)^2}+\frac{1}{(a_2+jb_2)^2}\right)+$$

$$+2\phi^{(5)}(t)\frac{\tau^5}{4^5 5!}\left(\frac{1}{(a_1+jb_1)^4}+\frac{1}{(a_2+jb_2)^4}\right)+...$$

By suitable selection of a_1, b_1, a_2, and b_2, some odd derivatives may disappear or be significantly reduced. For example, it is easy to see that for a_1=-a_2=b_1=b_2 all the derivatives of order 4n-1, n=1,2,..., will vanish.

The corresponding TF distribution is defined as:

$$GCD_{N=4}(t,\omega)\,|_{a_i,b_i}\;=$$

$$=\int_{-\infty}^{\infty}\prod_{i=1}^{2} s^{\pm(a_i+jb_i)}\left(t\pm\frac{\tau}{4(a_i+jb_i)}\right)e^{-j\omega\tau}d\tau. \quad (12)$$

Observe that in the case (a_1,b_1,a_2,b_2) = (1,0,0,1) (Fig 1.d), a form of the complex-lag distribution (N=4) [8] is obtained:

$$GCD_{N=4}(t,\omega)=$$

$$=\int_{-\infty}^{\infty} s(t+\frac{\tau}{4})s^{-1}(t-\frac{\tau}{4})$$

$$\times s^{-j}(t+j\frac{\tau}{4})s^{j}(t-j\frac{\tau}{4})e^{-j\omega\tau}d\tau. \quad (13)$$

Its spread factor is given in Table I (3^{rd} row). In addition, it is interesting to observe: (a_1,b_1,a_2,b_2)=$(1/2,\sqrt{3}/2,\sqrt{3}/2,1/2)$. By using the previous modification $e^{j\cdot angle(M(t,\tau))}$, we will obtain the spread factor of (13) divided by 2. Also, an additional advantage of using the points with smaller imaginary part is in reducing the miscalculations that may appear in the numerical realization of a signal with complex-lag argument [8].

For signals with higher non-stationary variations of IF, the precision can be additionally improved by using the multiple successive integrations approach. Thus, similarly as in (10), the L-form of (12) can be used. For instance, the L-form of $GCD_{N=4}(t,\omega)$ is defined as:

$$GCD_{N=4}^{L}(t,\omega)=$$

$$=\int_{-\infty}^{\infty} s^{L}(t+\frac{\tau}{4L})s^{-L}(t-\frac{\tau}{4L})$$

$$\times s^{-jL}(t+j\frac{\tau}{4L})s^{jL}(t-j\frac{\tau}{4L})e^{-j\omega\tau}d\tau. \quad (14)$$

The terms within the spread factor for $GCD_{N=4}^{L=2}(t,\omega)$ are reduced with respect to $GCD_{N=4}(t,\omega)$ (Table I). Regarding the realization complexity, the L-form of complex-lag distribution (for even L) can be easily obtained by performing the recursive realization as follows:

$$GCD_{N=4}^{L}(t,\omega)=$$

$$=\frac{1}{\pi}\int_{-\infty}^{\infty} GCD_{N=4}^{L/2}(t,\omega+\theta)GCD_{N=4}^{L/2}(t,\omega-\theta)d\theta. \quad (15)$$

This form is very suitable for practical applications. The influence of noise on $GCD_{N=4}^{L}(t,\omega)$ is analyzed in the next section.

By considering more points on the circle, the accuracy of IF estimation increases. However, the realization complexity also increases. Ideally, for an arbitrarily high number of points N

and number of integrations L, the spread factor tends to zero. The corresponding L-form of the N-th order complex-lag distribution can be defined as follows:

$$GCD_N^L(t,\omega) = \\ = \int_{-\infty}^{\infty} \prod_{i=1}^{N/2-1} s^{\pm L(a_i+jb_i)}\left(t \pm \frac{\tau}{N \cdot L(a_i+jb_i)}\right) e^{-j\omega\tau} d\tau. \quad (16)$$

This form offers new possibilities related to the selection of parameters a_i and b_i, which could be further explored in some future work. An interesting case is obtained for the equidistant points: $\{\pm(a_i+jb_i)\,|i=1,...,N/2-1\} = \{e^{j2\pi k/N}\,|k=1,...,N-1\}$. Note that for L=1, it corresponds to the generalized complex-lag TF distribution [9]. It can be seen from Table I (last row) that the distribution spread factor can be arbitrarily reduced by the suitable selection of parameters L and N (distribution order).

III. Influence of noise on the IF estimation

Consider a discrete signal $s(n) = re^{j\phi(n)}$(r is a signal amplitude, while $\phi(n)$ is a phase) corrupted by a Gaussian white noise $\nu(n)$. Without loss of generality, the discrete complex-lag distribution N=4, can be written as [8]:

$$GCD_{N=4}(n,\omega) = \\ = \sum_{m=-N_s/2}^{N_s/2-1} w(m)(s(n+m)+\nu(n+m)) \\ \times(s^*(n-m)+\nu^*(n-m)) \\ \times e^{-j2\ln\left|\frac{s(n+jm)+\nu(n+jm)}{s(n-jm)+\nu(n-jm)}\right|} e^{-j4m\omega}, \quad (17)$$

where m is a discrete lag coordinate, N_s is the number of signal samples, while $w(m)$ is a window. If the small noise is assumed, the following approximation can be used:

$$\ln|s(n+jm)+\nu(n+jm)| = \\ = \ln|s(n+jm)| + \Delta\theta_\nu(n,m), \quad (18)$$

where $\Delta\theta_\nu = Re\{\nu(n+jm)/s(n+jm)\}$ is a phase deviation of the terms with complex-lag argument. The values of signal with complex-lag argument are calculated using the analytical extension as follows [8]:

$$s(n+jm) = \\ = \sum_{k=k(n)-N_k}^{k=k(n)+N_k} STFT_s(n,k)e^{2\pi/N(n+jm)k}, \quad (19)$$

where $STFT$ denotes the short time Fourier transform, $k(n)$ is the position of transform $STFT_s(n,k)$ maximum at a given instant n, while the width of the signal component in the TF plane is $2N_k$+1. The same relation holds for the noise $\nu(n+jm)$. The bias and the variance of IF estimate for$GCD_{N=4}(n,\omega)$are derived as [8]:

$$bias\{\Delta\omega\} = \frac{1}{M_2}\left(\phi^{(5)}\frac{M_6}{5!} + \phi^{(9)}\frac{M_{10}}{9!} + ...\right),$$

$$var\{\Delta\omega\} \sim \\ \sim \frac{\sigma_\nu^2 N_w^3}{24r^2M_2^2}\left(\frac{1}{4} + \frac{3}{8\pi N_w}e^{2\pi N_w N_k/N_s}\right), \quad (20)$$

where the rectangular window $w(m)$ of width N_w is used, while $M_k = \sum_{m=-N_s/2}^{N_s/2-1} m^k w(m)$, $k=2,3,4,...$. Although $GCD_{N=4}(n,\omega)$ is of higher order N, comparing to the Wigner distribution, for the values: N_s=128, N_w=16, N_k =8, the variance in $GCD_{N=4}$ is of the same order as the variance in the Wigner distribution, while the bias is lower for several orders, providing a significant improvement of the IF estimation [8]. For the window length N_w that is approximately higher than 40, the representation starts to be sensitive to additive noise (the exponential term in (20) starts to be dominant). Observe that the variance decreases by decreasing the value of N_k. It is important to note that the value N_k will not influence bias as long as the signal component fits within the interval [$k(n)$-N_k,$k(n)$+N_k]. Thus, the optimal value N_k is determined by the signal component width. This interval can be automatically determined for each signal component (more details could be found in [8], pp. 482, 483).

In the sequel we provide analysis of noise influence in the case of L-complex-lag distribution (N=4). For a noisy signal $x(n)$, under the low-noise assumption, we can use the following model with additive noise:

$$x^L(n) = (s(n)+\nu(n))^L \cong$$
$$\cong s^L(n)+Ls^{L-1}(n)\nu(n) = s^L(n)+\xi(n), \quad (21)$$

where the autocorrelation function of $\xi(n)$ is defined as: $R_{\xi\xi}(n) = r^{2L-2}\sigma_\nu^2 L^2\delta(n)$, while σ_ν is the variance of noise $\nu(n)$. Therefore, the discrete form of L-complex-lag distribution for noisy signal can be defined as:

$$GCD_{N=4}^{L} =$$
$$\sum_{m=-N_s/2}^{N_s/2-1} w(m)(s^L(n+\frac{m}{L})+\xi(n+\frac{m}{L}))$$
$$\times((s^{*L}(n-\frac{m}{L})+\xi^*(n-\frac{m}{L})) \quad (22)$$

where: $\theta(n,m) = \ln\left|s(n+j\frac{m}{L})+\nu(n+j\frac{m}{L})\right|$. The bias and variance of the IF estimate are obtained as follows:

$$bias\{\Delta\omega\} = \frac{1}{M_2}\left(\phi^{(5)}\frac{M_6}{5!L^4}+\phi^{(9)}\frac{M_{10}}{9!L^8}+...\right),$$

$$\mathrm{var}\{\Delta\omega\} = \frac{\mathrm{L}^2\sigma_\nu^2}{8\mathrm{r}^2\mathrm{M}_2^2}\left(\left(\sum_{\mathrm{m}=-\mathrm{N_s}/2}^{\mathrm{N_s}/2}\mathrm{m}^2\mathrm{w}^2(\mathrm{m})\right)+\right.$$
$$\left.+\frac{1}{Nr^{2L-2}}\sum_{k=-N_k/L}^{N_k/L}\left(\sum_{m=-N_s/2}^{N_s/2} mw(m)e^{-2\pi km/N}\right)^2\right) \quad (23)$$

In the case of rectangular window $w(m)$ of length N_w, we have:

$$var\{\Delta\omega\} \sim \frac{L^2\sigma_\nu^2 N_w^3}{24r^2M_2^2}\times$$
$$\times\left(\frac{1}{4}+\frac{3}{8\pi r^{2L-2}N_w}e^{2\pi N_w N_k/N_s/L}\right). \quad (24)$$

If the second term within the brackets is small enough, the variance in the L-complex-lag distribution (N=4) is of the same order as the variance in the L-Wigner distribution (N=2). For example, $3\exp(2\pi N_w N_k/N_s/L)/8\pi r^{2L-2}N_w = 0.17$ holds for N_s=128, N_w=16, N_k=8, r=1, L=2. Therefore, the variance of the IF estimation based on the L-complex-lag distribution (N=4) is preserved at the same level as in the case of L-Wigner distribution [11], while the bias is significantly reduced.

IV. Examples

Consider the signal with fast varying IF:

$$x(t) = \exp(j\cdot(2\cos(2\cdot\pi\cdot t)+1/2\cos(6\cdot\pi\cdot t)$$
$$+1/2\cdot\cos(4\cdot\pi\cdot t)))+\nu(t),$$

where $\nu(t)$ is complex white Gaussian noise (SNR=30dB). The time interval $t\in[-1,1]$, with $\Delta t = 2/128$ is used. The results for Wigner distribution, L-Wigner distribution, Smoothed pseudo Wigner distribution (SPWD), $GCD_{N=4}$, $GCD_{N=4}^{L=2}$, and $GCD_{N=6}$ are given in Fig 2. Observe that the Wigner distribution, the L-Wigner distribution and the SPWD are useless for the IF estimation in this case. Namely, since the Wigner distribution is not able to follow the variations of IF, smoothing with respect to the frequency or both the time and frequency axes (in the cases of the L-Wigner and the SPWD, respectively) cannot produce satisfying results. Note that for the considered signal, $GCD_{N=4}$ follows the variations of IF (Fig 2.d), but the resolution is still not satisfactory. Thus, without affecting the complexity of realization, $GCD_{N=4}^{L=2}$ (Fig 2.e) improves the concentration of $GCD_{N=4}$ by reducing the inner-interferences. In comparison with $GCD_{N=6}$, $GCD_{N=4}^{L=2}$ is less sensitive to noise. Additionally, a multicomponent signal with fast varying IF is considered:

$$y(t) =$$
$$= exp(j\cdot(4.5\cdot\cos(\pi\cdot t)+3/4\cos(6\cdot\pi\cdot t)+8\cdot\pi\cdot t))+$$
$$+\exp\left(j\cdot(2\cdot\cos(\pi\cdot t)+2/3\cos(6\cdot\pi\cdot t)+\right.$$
$$\left.+\cos(2\cdot\pi\cdot t)-8\cdot\pi\cdot t)\right).$$

Time-frequency representations obtained by using the Wigner distribution, $GCD_{N=4}$ and its L-form are shown in Fig 2. f, g, and i, respectively. Again, the Wigner distribution is useless, while the $GCD_{N=4}$ can follow the

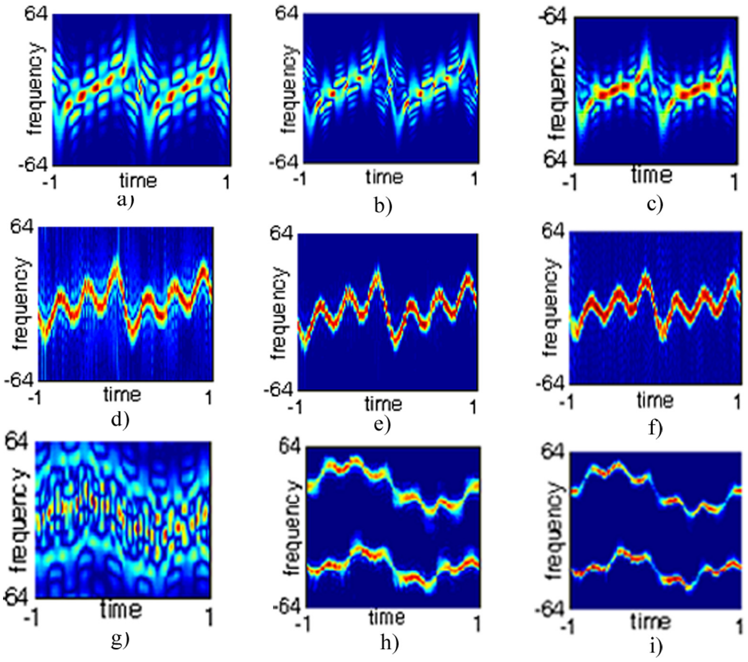

Fig. 2. Monocomponents signal x(t): a) Wigner distribution, b) L-Wigner distribution, c) SPWD, d) $GCD_{N=4}$, e) $GCD_{N=4}^{L=2}$, f) $GCD_{N=6}$, Multicomponent signal y(t): g) Wigner distribution, h) $GCD_{N=4}$, i) $GCD_{N=4}^{L=2}$

variations of IF, but the concentration should be improved. It is done by using its L-form i.e. $GCD_{N=4}^{L=2}$ (Fig 2.i). The cross-terms free realization for multicomponent signals is straightforward, according to the procedure given in [8].

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