based detector discussed in this paper, will perform better than any other detector, since it uses all available prior information. When only the time-bandwidth product and the relative bandwidth of the signal are known, then the detector of [13] should be used. Use of that detector in our case is also possible at some performance degradation resulting from not using prior information concerning exact waveshape. The amount of performance loss resulting from disregarding prior information in the detector of [13], as well as comparison of the performance of the two detectors when the assumed waveshape is actually distorted, are still open questions.

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On the Wigner Distribution of Discrete-Time Noisy Signals with Application to the Study of Quantization Effects

LJubiša Stanković and Srdjan Stanković

Abstract— **The influence of noise to the Wigner distribution (WD) of discrete-time signals is analyzed. The signals contaminated by white noise are considered. The expressions for the mean and variance of WD estimator are derived. It is then shown that the application of only one window is sufficient to make the variance of estimator finite. That is a significant difference from the analog case, where two windows are needed. The expression for the optimal window width is derived. The results are illustrated by several numerical examples. Using the derived expressions, the effects of finite register length to the WD are analyzed. Fixed and floating point arithmetics are considered.**

I. INTRODUCTION

The Wigner distribution, as a tool for signal analysis and synthesis of time-varying systems, has attracted the attention of many researches during the last decade. A great number of papers have addressed the theoretical problems as well as the application of this type of distribution. We refer the readers to the review papers [1], [2].

The influence of noise to the WD of analog signals is treated in [1], [3]. This paper extends the analysis to the case of discrete-time signals. We show that the difference from the analog case is not only formal. In the discrete-time domain case one truncation window is sufficient, whereas in the analog case it was not. The expression for optimal window width is derived. The obtained results are demonstrated on the numerical examples with frequency modulated (FM) signals. The expressions describing the effects of finite register length are given. Fixed and floating point arithmetics are considered.

II. THEORY

The Wigner distribution of a discrete-time signal $f(n)$ is defined as

$$W_{ff}(n,\theta) = a \sum_{k=-L}^{L} f(n+k)f^*(n-k)e^{-j2\theta k} \tag{1}$$

where $a = 2$. Without loss of generality, we will simplify the presentation by taking $a = 1$. Taking $L \to \infty$, one may include the case of long signals. If the signal is contaminated by additive noise, then $f(n)$ from (1) should be replaced by $x(n) = f(n) + \nu(n)$, where the noise is represented by $\nu(n)$.

We will consider the case when signal $f(n)$ is deterministic and the noise falls into one of the following categories: real, complex or analytic. The modifications for random Gaussian signals are given.

A. The Real Noise Case

We will consider a deterministic signal $f(n)$ with white Gaussian noise $\nu(n)$. The variance of the noise is assumed to be σ_ν^2. The mean

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of WD of the signal $x(n) = f(n) + \nu(n)$ is

$$E\{W_{xx}(n,\theta)\} = W_{ff}(n,\theta) + \sum_{k=-L}^{L} R_{\nu\nu}(2k)e^{-jk2\theta} = W_{ff}(n,\theta) + \sigma_\nu^2 \quad (2)$$

where $R_{\nu\nu}(k)$ is the autocorrelation of noise $\nu(n)$. We see that $W_{xx}(n,\theta)$ is biased.

The variance of WD estimator $W_{xx}(n,\theta)$ is

$$\sigma_{xx}^2 = E\{W_{xx}(n,\theta)W_{xx}^*(n,\theta)\} - E\{W_{xx}(n,\theta)\}E\{W_{xx}^*(n,\theta)\}.$$

After some straightforward manipulations we get two components of variance $\sigma_{xx}^2 = \sigma_{f\nu}^2 + \sigma_{\nu\nu}^2$, one ($\sigma_{f\nu}^2$) depending on both signal and noise, and the other ($\sigma_{\nu\nu}^2$) depending on the noise only (see the Appendix). For the white Gaussian noise we get

$$\sigma_{f\nu}^2 = \sigma_\nu^2 \sum_{k=-L}^{L} [|f(n+k)|^2 + |f(n-k)|^2 + 2\mathrm{Real}\{f^2(n+k)e^{-jk4\theta}\}] \quad (3)$$

$$\sigma_{\nu\nu}^2 = \sigma_\nu^4 \sum_{k=-L}^{L} [1 + e^{-jk4\theta}] = \sigma_\nu^4 N\left[1 + \frac{\sin(2\theta N)}{N\sin(2\theta)}\right], \qquad N = 2L+1. \quad (4)$$

If the noise is white, but not Gaussian, the results are slightly different[1] (see the Appendix). Therefore, we will not pursue here those differences.

For the finite power signals, $|f(k)|^2 \le P_f$, we have

$$\sigma_{xx}^2 \le \sigma_\nu^2 N(4P_f + 2\sigma_\nu^2). \quad (5)$$

For FM signals, $f(k) = e^{j\phi(k)}, |f(k)|^2 = P_f = 1$, so the previous expression is close to the exact one.

From the above expressions we see that the variance of the estimator $W_{xx}(n,\theta)$ approaches infinity, as $L \to \infty$. The truncation window is necessary and, as it is clear from (5), is also sufficient to make the variance finite. Let us consider the window $w(n)$ whose width is $N = 2L+1$. The WD of the truncated signal (the Pseudo Wigner distribution) is

$$W_{xx}(n,\theta) = \sum_{k=-\infty}^{\infty} w(k)w(-k)x(n+k)x^*(n-k)e^{-jk2\theta} \quad (6)$$

whose mean, after same modifications, becomes

$$E\{W_{xx}(n,\theta)\} = W_{ff}(n,\theta) *_\theta F_w(2\theta) + \sigma_\nu^2 *_\theta F_w(2\theta) \quad (7)$$

where $F_w(\theta) = FT[w(k)w(-k)]$ is the Fourier transform (FT) of the product $w(k)w(-k)$.

The window causes an increase in the WD bias. The second term on the right-hand side in (7) is constant, so one may assume it does not introduce any distortion to the WD. Using a Taylor series expansion of $W_{ff}(n,\theta)$, the first term can be approximated by

$$W_{ff}(n,\theta) *_\theta F_w(2\theta) = \frac{1}{\pi}\int_{-\pi/2}^{\pi/2} W_{ff}(n,\theta-\omega)F_w(2\omega)d\omega \cong W_{ff}(n,\theta) + \frac{1}{8}\frac{\partial^2 W_{ff}(n,\theta)}{\partial\theta^2}m_2. \quad (8)$$

The bias is

$$b = \frac{1}{8}\frac{\partial^2 W_{ff}(n,\theta)}{\partial\theta^2}m_2 = \frac{1}{8}b_f m_2 \quad (9)$$

where $m_2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}\omega^2 F_w(\omega)d\omega$ is the amplitude moment of the window $w(k)w(-k)$. If $w(k)w(-k)$ is the Hanning window, then $m_2 = 0.5(\pi/L)^2$.

The optimal window width can be obtained by minimizing the error defined as: $e^2 = b^2 + \sigma_{xx}^2$. If the window is not rectangular then the values of variance (5) should be modified as: $\sigma_{xx}^2 \le \sigma_\nu^2 E_w(4P_f + 2\sigma_\nu^2)$, where $E_w = \sum_{k=-L}^{L}[w(k)w(-k)]^2$ is the energy of the $w(k)w(-k)$ window. For the Hanning window ($E_w = 3N/8$)

$$e^2 = b_f^2(\pi/L)^4/256 + 3(2L+1)\sigma_\nu^2(4P_f + 2\sigma_\nu^2)/8. \quad (10)$$

From $\partial e^2/\partial L = 0$, the optimal window width follows

$$2L = \sqrt[5]{b_f^2\pi^4/[3\sigma_\nu^2(2P_f + \sigma_\nu^2)]}. \quad (11)$$

The bias is minimal for the minimum amplitude moment windows. If the energy E_w is constant then the error is minimal if a window is parabolic [4].

B. Complex Noise With Independent Real and Imaginary Parts

Assuming that the variances of the real and imaginary parts are equal to $\sigma_\nu^2/2$, the total noise variance is σ_ν^2. The components of the variance are

$$\sigma_{f\nu}^2 = \sigma_\nu^2 \sum_{k=-L}^{L} [|f(n+k)|^2 + |f(n-k)|^2]$$

$$\sigma_{\nu\nu}^2 = \sigma_\nu^4 \sum_{k=-L}^{L} 1 = \sigma_\nu^4 N. \quad (12)$$

For the FM signals ($f(k) = e^{j\phi(k)}, |f(k)|^2 = P_f = 1$) we have $\sigma_{xx}^2 = \sigma_\nu^2 N(2P_f + \sigma_\nu^2)$.

C. The Noise in the Form of an Analytic Signal

Recall that the analytic part of a signal $\nu(n)$ is defined by

$$\nu_a(n) = \nu(n) + j\nu_h(n) \quad (13)$$

where $\nu_h(n)$ is the Hilbert transform of the signal $\nu(n)$. The analytic signal is commonly used in the calculation of the WD. In that case the noise has real and imaginary parts which are related via a Hilbert transform, thus being correlated. If the spectral power density of the input noise is $\sigma_\nu^2/2$, the spectral power density of analytic noise is $S_{\nu a\nu a}(\theta) = 2\sigma_\nu^2 U(\theta)$ for $|\theta| < \pi$, where $U(\theta)$ is the unit step function.

Using Parseval theorem[2] and the results from the Appendix, we get

$$\sigma_{f\nu}^2 = \frac{2}{2\pi}\int_{-\pi}^{\pi}|F(e^{j\omega})|^2 S_{\nu a\nu a}(2\theta-\omega)d\omega = \frac{4}{2\pi}\sigma_\nu^2\begin{cases}\int_0^{2\theta}|F(e^{j\omega})|^2 d\omega & 0 \le \theta < \pi/2\\ \int_{\pi+2\theta}^{\pi}|F(e^{j\omega})|^2 d\omega & -\pi/2 \le \theta < 0\end{cases} \le 4E_f\sigma_\nu^2. \quad (14)$$

The component of variance, depending on the noise only, is

$$\sigma_{\nu\nu}^2 = \sum_{k_1=-L}^{L} S_{\nu a\nu a}(2\theta) * S_{\nu a\nu a}(2\theta) = 4\sigma_\nu^4 \sum_{k_1=-L}^{L}|\theta/\pi| = 4\sigma_\nu^4 N|\theta/\pi|, |\theta| < \pi/2. \quad (15)$$

We can see that both components are strongly dependent on θ and equal to zero for $\theta = 0$. Besides, $\sigma_{f\nu}^2$ is a nondecreasing

[1] For white uniformly distributed noise, the variance is

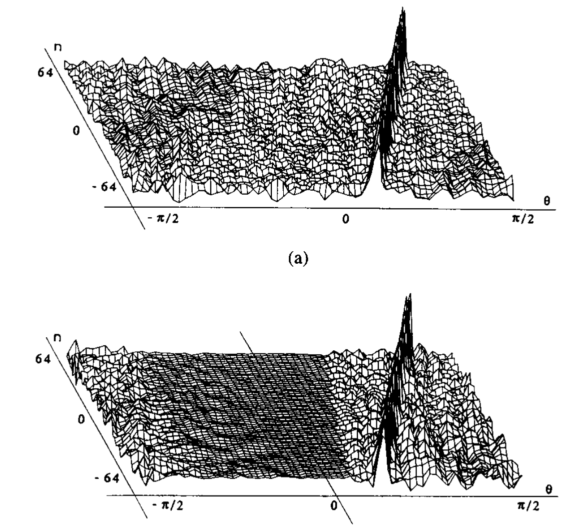
$$\sigma_{\nu\nu}^2 = \sigma_\nu^4 N\left\{1 - \frac{6}{5N} + \frac{\sin[2\theta N]}{N\sin(2\theta)}\right\} \cong \sigma_\nu^4 N\left\{1 + \frac{\sin[2\theta N]}{N\sin(2\theta)}\right\}.$$

[2] Parseval theorem:

$$\sum_{k=-\infty}^{\infty} x(k)y^*(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega.$$

TABLE I
THE VARIANCE OF THE WIGNER DISTRIBUTION ESTIMATOR FOR RANDOM WHITE SIGNALS

Real signal and noise	$\sigma_{xx}^2 = (\sigma_\nu^2 + \sigma_f^2)^2 N\left[1 + \frac{\sin(2\theta N)}{N\sin(2\theta)}\right]$
Complex signal and noise	$\sigma_{xx}^2 = (\sigma_\nu^2 + \sigma_f^2)^2 N$
Analytic part of the signal and noise	$\sigma_{xx}^2 = 4(\sigma_\nu^2 + \sigma_f^2)^2 N\lvert\theta/\pi\rvert$ $\overline{\sigma}_{xx}^2 = (\sigma_\nu^2 + \sigma_f^2)^2 N$; mean value over θ



(b)

Fig. 1. (a) Wigner distribution of a linear frequency modulated signal with Gaussian noise. (b) Analytic signal and noise; Hanning window $w(k)w(-k)$ is used with $L = 32$.

function, where θ ranges from 0 to $\pi/2$ and from 0 to $-\pi/2$. The variance reaches maximum for $|\theta| = \pi/2$, the maximum values being $\sigma_{xx}^2 = 4E_f\sigma_\nu^2 + 2N\sigma_\nu^4$, where E_f is the energy of analytic signal $f(n)$.

D. Random White Signal

If the signal $f(n)$ is random, then the real advantage of the WD is realized only when the signal is nonstationary [6]. The aim of this section is to provide background for a study of the influence of quantization effects to the WD. Only this simple stationary case allows one to obtain closed form results [5].

If σ_f^2 is the variance of the signal $f(n)$, then the values of σ_{xx}^2 may be obtained from the expression for $\sigma_{\nu\nu}^2$ in all three considered cases (4), (12), (15), replacing σ_ν^2 by $\sigma_f^2 + \sigma_\nu^2$. The results are presented in Table I.

III. NUMERICAL EXAMPLE

Consider linearly modulated signals

$$\begin{aligned} f_1(n) &= e^{j4\pi(n/64+2)^2} + \nu(n) \\ f_2(n) &= \cos[4\pi(n/64+2)^2] + \nu(n). \end{aligned} \tag{16}$$

The noise is assumed to be white, Gaussian with $\sigma_\nu^2 = 1/4$. The Wigner distributions of the noisy signal $f_1(n)$ and of the analytic version of the noisy signal $f_2(n)$ are shown in Fig. 1(a) and 1(b), respectively.

The above graphics exhibit complete agreement with the previously derived results. We can see that for the analytic signal the variance of WD is zero for $\theta = 0$ and increases as $|\theta|$ increases. Noise is present at the negative frequencies, as expected from eqns. (14) and (15). From Fig. 1(a) and 1(b) we see that the distribution of noise is quite different depending on the way of the WD calculation.

IV. QUANTIZATION—THE EFFECTS OF FINITE LENGTH REGISTER

An interesting kind of noise is the one resulting from the quantization of a signal or the results of certain mathematical operation.

For the ensuing analysis we will make the following usual assumptions: 1) The length of registers is $b+1$ bits (b bits for absolute value and one sign bit); 2) Each of the noises is white with uniform distribution in the interval from $-2^{-b}/2$ to $2^{-b}/2$; 3) The mean of such a noise is zero and the variance is $\sigma_B^2 = 2^{-2b}/12$; 4) The quantization noises are mutually uncorrelated, and they are not correlated with the signals.

A. Fixed-Point Arithmetic

Initially suppose that the values of the signal are such that no overflow will occur. The discrete Wigner distribution, by definition, is

$$W_{ff}(n,m) = \sum_{k=-L}^{L} f(n+k)f^*(n-k)e^{-j\frac{4\pi}{N}km}. \tag{17}$$

The effects of quantization of the signal $f(n)$ can be modeled by the noise $\epsilon(n)$, so the WD is calculated for the signal $x(n) = f(n)+\epsilon(n)$. The WD model for analysis is

$$\begin{aligned} W(n,m) &= \sum_{k=-L}^{L} \{[x(n+k)x^*(n-k) + e(n,k)]e^{-j\frac{4\pi}{N}km} \\ &\quad + \mu(n,k,m)\} \\ &= W_{xx}(n,m) + \sum_{k=-L}^{L} [e(n,k)e^{-j\frac{4\pi}{N}km} + \mu(n,k,m)] \end{aligned}$$

where $e(n,k)$ and $\mu(n,k,m)$ are noises due to the quantization of $x(n+k)x^*(n-k)$, and its product with the basis functions $e^{-j4\pi km/N}$. In the last expression we have taken into account all the quantization errors except the one of the basis functions. This error has some deterministic properties, although it can be modeled as white noise [5].

The mean and variance of the Wigner distribution are

$$E\{W_{xx}(n,m)\} = W_{ff}(n,m) + \sigma_\epsilon^2, \quad \sigma^2 = \sigma_{xx}^2 + N\sigma_\epsilon^2 + N\sigma_\mu^2 \tag{18}$$

where σ_{xx}^2 is the variance when only the input noise $\epsilon(n)$ exists, and the arithmetic is ideal (Table I with $\theta = 2\pi m/N$), $\sigma_{xx}^2 = N(\sigma_f^2 + \sigma_\epsilon^2)^2$, with $2\sigma_\epsilon^2 = \sigma_\epsilon^2 = \sigma_\mu^2 = 4\sigma_B^2$.

The noise to signal ratio (NSR) will be defined as

$$\text{NSR} = \frac{\sigma^2 - \sigma^2_{|\text{without any noise}}}{\sigma^2_{|\text{without any noise}}}. \tag{19}$$

For the white complex signal with the variance σ_f^2, we have

$$\text{NSR} = (2\sigma_\epsilon^2\sigma_f^2 + \sigma_\epsilon^4 + \sigma_\epsilon^2 + \sigma_\mu^2)/\sigma_f^4. \tag{20}$$

The direct calculation of DFT, by its definition, is not used in practice, but for fixed-point arithmetics, the same results are valid for the FFT algorithms. For example, for decimation-in-time FFT algorithm the variance term $N\sigma_\mu^2$ in (18) becomes $(N-1)\sigma_\mu^2$, [5].

If the values of a signal are not small we have to ensure that no overflow occurs during the calculation of the FT. When the signal is

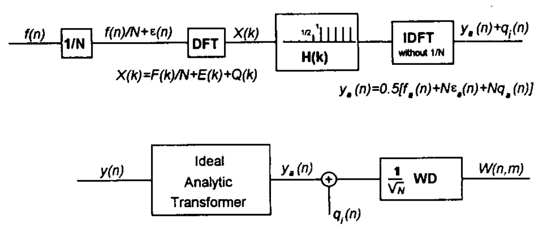


Fig. 2. Analytic signal calculation flow graph.

uniformly distributed over [0, 1), we can be sure that the overflow is avoided if we use one of the following procedures: 1) division of the input signal by $\sqrt{N}$, 2) use of the FFT algorithms with factors 1/2, [5], or 3) use of the floating-point arithmetics.

1) When the product in Wigner distribution is divided by N, the signal $f(n)$ is divided by $\sqrt{N}$, so the overflow will not occur in any case. The NSR is

$$\mathrm{NSR} = (2\sigma_\epsilon^2\sigma_f^2 + \sigma_\epsilon^4 + N^2\sigma_\epsilon^2 + N^2\sigma_\mu^2)/\sigma_f^4. \tag{21}$$

2) An alternative way to avoid the overflow is the use of the factors 1/2 in FFT butterflies. Using those algorithms, [5], and (20), we get

$$\mathrm{NSR} = (2\sigma_\epsilon^2\sigma_f^2 + \sigma_\epsilon^4 + \sigma_\epsilon^2 + 4N\sigma_\mu^2)/\sigma_f^4. \tag{22}$$

We see that the dominant term is of order N, unlike N^2 in (21).

B. Analytic Signals and Fixed-Point Arithmetic

An often used procedure to obtain analytic part of a real signal is shown in Fig. 2. The variance of Wigner distribution estimator can be obtained using previous expressions. For white signals we get

$$\sigma^2 = \sigma_{xx}^2 + \sigma_{q_iq_i}^2 + N\sigma_\epsilon^2 + N\sigma_\mu^2$$

with

$$\begin{aligned}\sigma_{xx}^2 &\cong N|\theta/4\pi|[\sigma_f^4/N^2 + 2\sigma_f^2(\sigma_\epsilon^2+\sigma_q^2) + N^2(\sigma_\epsilon^2+\sigma_q^2)^2]\\ \theta &= 2\pi m/N.\end{aligned}$$

The mean of the NSR for $|\theta| < \pi/2$ is

$$\begin{aligned}\mathrm{NSR} = [&2N^2(\sigma_\epsilon^2+\sigma_q^2)\sigma_f^2 + N^4(\sigma_\epsilon^2+\sigma_q^2)^2 + 16N^4\sigma_{qi}^4\\ &+16N^2(\sigma_\epsilon^2+\sigma_\mu^2)]/\sigma_f^4\end{aligned} \tag{23}$$

with: $4\sigma_\epsilon^2 = \sigma_\mu^2 = \sigma_\epsilon^2 = \sigma_q^2 = \sigma_{qi}^2 = 4\sigma_B^2$. We emphasize that (23) is the mean of the NSR, and that NSR is dependent on frequency. The frequency dependence is in accordance with Fig. 1(b).

C. Floating-Point Arithmetic

When floating-point arithmetics is used, the influence of finite length register can be represented by multiplicative noise [5], i.e., $Q[x_1x_2] = x_1x_2(1+\epsilon)$, where ϵ is an error uniformly distributed over the interval $-2^{-b} < \epsilon < 2^{-b}(\sigma_B^2 = 2^{-2b}2/3)$, for which earlier made assumptions (in the introductionary part of section 4.) are valid. In this case additions also produce noise which can be represented by multiplicative noise.

A model for the mean and variance calculation is

$$\begin{aligned}W(n,m) = \sum_{k=-L}^{L-1}\{&x(n+k)x^*(n-k)[1+e(n,k)]e^{-j\frac{4\pi}{N}km}\\ &\times[1+\mu(n,k,m)]\prod_{p=1}^{L_p}[1+g(n,k,m,p)]\}\end{aligned} \tag{24}$$

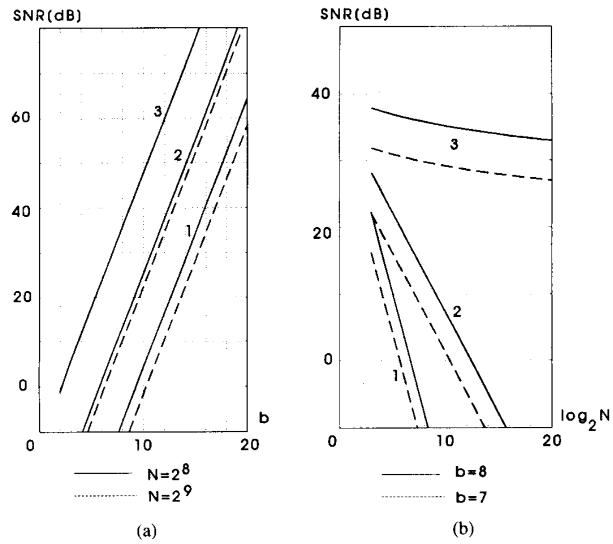


Fig. 3. The SNR = 1/NSR for various: (a) register lengths b; (b) lengths of signal N. "1": Fixed-point arithmetics with DFT or FFT. "2": Fixed point arithmetics with FFT and factors 1/2. "3": Floating-point arithmetics.

where $Lp = \log_2(2L)$. The additions are performed by adding adjacent elements in the first step, then the adjacent sums in the next steps. This corresponds to the butterflies in the FFT algorithm.

Since the errors are small, we will neglect all higher order error terms, so we get

$$\prod_{p=1}^{L_p}[1+g(n,k,m,p)] \cong 1 + \sum_{p=1}^{Lp} g(n,k,m,p). \tag{25}$$

After relatively straightforward transformations, for white signals, we get

$$\mathrm{NSR} = 2\sigma_\epsilon^2/\sigma_f^2 + (Lp+2)\sigma_c^2, \qquad 2\sigma_\epsilon^2 = \sigma_c^2 = 4\sigma_B^2. \tag{26}$$

We can see that the NSR depends on $\log_2(N)$, which is a much better result than the one obtained with fixed-point arithmetics (20)–(23). The SNR=1/NSR for various cases is shown in Fig. 3(a) and 3(b).

V. Conclusion

The analysis of the influence of noise on the Wigner distribution of discrete-time signals is performed. Simple expressions for the mean and variance are derived. It is shown that, in the case of discrete-time signals, only one window is sufficient to make the variance finite. The optimal window width is derived. Complex and analytic noises are considered. The obtained results are used in the analysis of quantization effects.

Appendix

The variance components are

$$\begin{aligned}\sigma_{f\nu}^2 = \sum_{k_1=-L}^{L}\sum_{k_2=-L}^{L}\{&f(n+k_1)f^*(n+k_2)R_{\nu\nu}(k_1-k_2)\\ &+f(n+k_1)f(n-k_2)R_{\nu^*\nu}(-k_1-k_2)\\ &+f^*(n-k_1)f(n-k_2)R_{\nu\nu}(k_1-k_2)\\ &+f^*(n-k_1)f^*(n+k_2)R_{\nu\nu^*}(k_1+k_2)\}\\ &\times e^{-j2\theta(k_1-k_2)}\end{aligned} \tag{A1}$$

$$\sigma_{\nu\nu}^2 = \sum_{k_1=-L}^{L} \sum_{k_2=-L}^{L} E\{\nu(n+k_1)\nu^*(n-k_1)\nu^*(n+k_2) \times \nu(n-k_2)\}e^{-j2\theta(k_1-k_2)} - \sum_{k_1=-L}^{L} \sum_{k_2=-L}^{L} E\{\nu(n+k_1)\nu^*(n-k_1)\} \times E\{\nu^*(n+k_2)\nu(n-k_2)\} \times e^{-j2\theta(k_1-k_2)}. \tag{A2}$$

For the Gaussian noise $\nu(n)$ we have [4]

$$E\{\nu(n+k_1)\nu^*(n-k_1)\nu^*(n+k_2)\nu(n-k_2)\} = R_{\nu\nu}(2k_1)R_{\nu\nu}(-2k_2) + R_{\nu\nu}^2(k_1-k_2) + R_{\nu\nu^*}(k_1+k_2) R_{\nu^*\nu}(-k_1-k_2).$$

So, it is

$$\sigma_{\nu\nu}^2 = \sum_{k_1=-L}^{L} \sum_{k_2=-L}^{L} \{R_{\nu\nu}^2(k_1-k_2) + R_{\nu\nu^*}(k_1+k_2)R_{\nu^*\nu}(-k_1-k_2)\} \times e^{-j2\theta(k_1-k_2)}. \tag{A3}$$

For any other, real non-Gaussian, white noise

$$E\{\nu(n+k_1)\nu(n-k_1)\nu(n+k_2)\nu(n-k_2)\} = \sigma_\nu^4\delta(k_1-k_2) + \sigma_\nu^4\delta(k_1+k_2) + [E\{\nu^4(n)\} - 2\sigma_\nu^4]\delta(2k_1)\delta(-2k_2). \tag{A4}$$

For the Gaussian white noise $E\{\nu^4(n)\} = 3\sigma_\nu^4$, and for the uniform white noise $E\{\nu^4(n)\} = 9\sigma_\nu^4/5$.

For the analytic signal the variance components can be written in a modified version

$$\sigma_{f\nu}^2 = \sum_{k_1=-L}^{L} \sum_{k_2=-L}^{L} [f(n+k_1)f^*(n+k_2)R_{\nu_a\nu_a}(k_1-k_2) + f^*(n-k_1)f(n-k_2)R_{\nu_a\nu_a}(k_1-k_2)] \times e^{-j2\theta(k_1-k_2)}.$$

For $L \to \infty$ the previous expression can be understood as

$$\sigma_{f\nu}^2 = \sum_{k_1=-L}^{L} \{f(n+k_1)[f^*(n+k_1) *_{k_1} R_{\nu_a\nu_a}(k_1)e^{-j2\theta k_1}] + f^*(n-k_1)[f(n-k_1) *_{k_1} R_{\nu_a\nu_a}(k_1) \times e^{-j2\theta k_1}]\}$$

where $*_{k_1}$ denotes convolution along k_1.

The variance component, depending on the noise only, for analytic signal is given by

$$\sigma_{\nu\nu}^2 = \sum_{k_1=-L}^{L} \sum_{k_2=-L}^{L} R_{\nu_a\nu_a}^2(k_1-k_2)e^{-j2\theta(k1-k2)} \tag{A5}$$

which may be interpreted, for $L \to \infty$, as the sum of FT of the product $R_{\nu_a\nu_a}(k)R_{\nu_a\nu_a}(k)$, $k = k_1 - k_2$.

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A Reduced Parameter Bilinear Time Series Model

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***Abstract*— A new bilinear time series structure is proposed and is tested on three sample time series to demonstrate its effectiveness. It is found that the proposed bilinear model can represent both nonlinearity and multiperiodicity, and it therefore provides a useful model class for general applications. In addition, the proposed bilinear model uses fewer parameters than conventional bilinear models with the same structure.**

I. Introduction

The bilinear time series model, which is a special class of nonlinear model, has been proposed and studied extensively by Granger and Anderson [1], Priestly [2], Subba Rao and Gabr [3], and Mohler [4]. When compared with general nonlinear models (e.g., the Volterra series expansions), the bilinear model is nearly linear and therefore may be identified without excessive computational requirements. In addition, it can be shown [2] that the bilinear model can approximate to an arbitrary degree of accuracy any "well behaved" Volterra series relationship over a finite time interval. In view of this, the bilinear models represent a powerful class of nonlinear models that has been successfully applied to many real problems [3].

One of the disadvantages of the conventional bilinear model is that the number of parameters to be estimated is typically large. In this correspondence, a new form of bilinear model that can significantly reduce the number of parameters while maintaining a structure similar to the conventional bilinear model is considered. Another improvement over the conventional bilinear model is the ability to conveniently create a periodic or seasonal variation of the model.

Section II presents the reduced parameter bilinear model. Section III discusses identification and estimation techniques. Section IV presents some experimental results, and Section V provides a summary and conclusions.

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