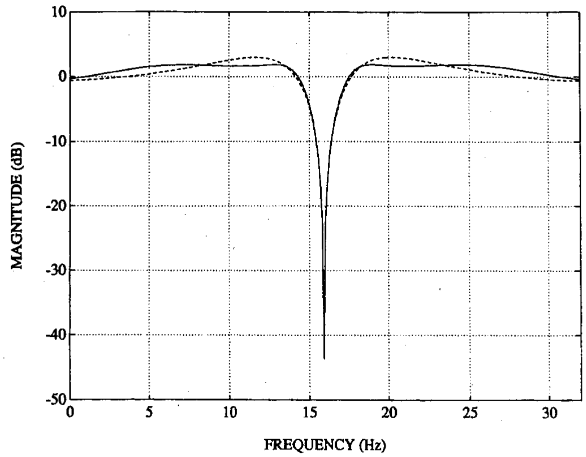

Fig. 8. Frequency response of FXLMS adaptive notch filter of Fig. 3 with optimized second-order compensator of Table IV.

ilar to the single-channel response except for the factor $F(\omega)$ that multiplies the single-channel open loop response $H(\omega)$. Standard control theory can be applied to determine the effect of $F(\omega)$ on the closed loop response. For example, the amplitude and phase of $F(\omega)$ will perturb of $H(\omega)$ on the Nyquist plot or Nichols chart, producing known effects. For $\alpha = 0$, $F(\omega) = 1$, and (22) reduces exactly to the single-channel response, as expected. Also note that for $\delta = 0$, $F(\omega) = |1 + \alpha|^2$ is a constant and thus (22) is identical to the single-channel closed loop response (2), except that the constant factor $|1 + \alpha|^2$ multiplies μ. In the region of the notch at $\omega \approx \omega_1$, it can be easily shown that

$$F(\omega) \approx |1 + \alpha e^{-j\omega_1\delta}|^2, \qquad |\omega - \omega_1| << |(1/\alpha + e^{-j\omega_1\delta})/\delta|. \tag{24}$$

Thus, under this condition $F(\omega)$ is nearly constant and again will only have the effect of increasing the effective gain.

The single-stage, multichannel results can be easily generalized to a multistage cascade by simply replacing all of the blocks in Fig. 4 with their matrix equivalents. Thus, C becomes the $L \times M$ cancellation path transfer function matrix C and each A_n becomes a multichannel version of the narrow-band FXLMS algorithm [6], [7]. An application example for a two-channel, three-stage vibration control problem can be found in [8].

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Wigner Distribution of Noisy Signals

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Abstract—**This correspondence presents an analysis of noise influence to the Wigner distribution. The mean and variance of Wigner distribution of signals contaminated by additive noise are derived. It is shown that in the case of white-noise, Wigner distribution, calculated by definition, cannot be used for estimation. For estimation it is not sufficient windowing in time domain, commonly used in signal processing. The smoothing in time domain is necessary. Very simple expressions for variance and bias estimation are obtained. The analytic signal and noise are considered. In that case, only one window can make variance finite. The results are demonstrated on numerical examples with linear and sinusoidal chirp pulses.**

I. Introduction

The Wigner distribution was first introduced in the field of quantum mechanics [1]. In signal processing it was used by Ville [2], so that it is often called Wigner–Ville distribution. This distribution belongs to the class of time-frequency signal representations, whose intensive research began last decade. The papers [3]–[5] had a significant stimulus to that. Problems of discretization and aliasing, as well as algorithms for efficient calculation of this distribution, are analyzed and described in [6]–[12]. The applications of Wigner distribution are various: analysis of nonstationary signals [13], radar signals [14], biomedical signals [15], analysis and synthesis of time varying filters [16], [17], and image processing [18], [19].

The influence of noise to the Wigner distribution, and its modifications, is analyzed in this correspondence. On the basis of this analysis, a modification of a Wigner distribution that can be used as an estimator with a finite variance is pointed out. That modification, in the cases of noisy signals, will not only improve results, as has been done in [18] for artifacts that appears in Wigner distribution of signals without noise, but will play an essential role in making the variance finite. For analytic signal and noise it is shown that only the truncation window can be sufficient to make variance finite. Examples with linear and sinusoidal frequency modulated signals and Gaussian white noise (real and "analytic") are given as illustrations.

II. Theory

We shall consider a deterministic signal $f(t)$ contaminated by zero-mean real stationary noise $n(t)$, given by expression

$$x(t) = f(t) + n(t). \tag{1}$$

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Boldfaced letters denote random variables. The Wigner distribution of signal $\boldsymbol{x}(t)$ is, by definition

$$W_{xx}(\omega,t) = \int_{-\infty}^{\infty} x(t + \tau/2)x^* (t - \tau/2)e^{-j\omega\tau}\, d\tau. \tag{2}$$

Substituting $\boldsymbol{x}(t)$ by $f(t) + \boldsymbol{n}(t)$, we get

$$W_{xx}(\omega, t) = W_{ff}(\omega, t) + W_{nn}(\omega, t) + W_{fn}(\omega, t) + W_{nf}(\omega; t). \tag{3}$$

The mean of Wigner distribution is

$$E\{W_{xx}(\omega, t)\} = W_{ff}(\omega, t) + S_n(\omega) \tag{4}$$

because $E\{W_{nn}(\omega, t)\} = S_n(\omega)$ and $E\{W_{fn}(\omega, t)\} = E\{W_{nf}(\omega, t)\} = 0$. The Wigner distribution of noisy signals has the bias equal to the noise power density function $S_n(\omega)$.

The variance of random variable $W_{xx}(\omega, t)$ is

$$\sigma_{xx}^2 = E\{|W_{xx}(\omega, t) - E\{W_{xx}(\omega, t)\}|^2\}. \tag{5}$$

The mods of variance, which are not equal to zero, will be separated in two parts: one depending on the signal and noise σ_{fn}^2, and the other depending on the noise only σ_{nn}^2.

$$\begin{aligned}\sigma_{fn}^2 &= E\{W_{fn}(\omega, t)\, W_{fn}^*(\omega, t)\} + E\{W_{nf}(\omega, t)\, W_{nf}^*(\omega, t)\} \\ &\quad + E\{W_{fn}(\omega, t)\, W_{nf}^*(\omega, t)\} + E\{W_{nf}(\omega, t)\, W_{fn}^*(\omega, t) \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left\{\left[f\left(t + \frac{\xi}{2}\right)f^*\left(t + \frac{\zeta}{2}\right) + f^*\left(t - \frac{\xi}{2}\right)\right.\right. \\ &\quad \left.\cdot f\left(t - \frac{\zeta}{2}\right)\right] R_n\left(\frac{\xi - \zeta}{2}\right) \\ &\quad + \left[f^*\left(t - \frac{\xi}{2}\right)f^*\left(t + \frac{\zeta}{2}\right) + f\left(t + \frac{\xi}{2}\right)f\left(t - \frac{\zeta}{2}\right)\right] \\ &\quad \left.\cdot R_n\left(\frac{\xi + \zeta}{2}\right)\right\} e^{-j\omega(\xi - \zeta)}\, d\xi\, d\zeta. \end{aligned} \tag{6}$$

For the second part of variance $\sigma_{nn'}^2$, we will first determine

$$\begin{aligned}E\{|W_{nn}(\omega, t)|^2\} &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E\{n(t + \xi/2)n\,(t - \xi/2) \\ &\quad \cdot n\,(t + \zeta/2)n\,(t - \zeta/2)\}\, e^{-j\omega(\xi - \zeta)}\, d\xi\, d\zeta. \end{aligned} \tag{7}$$

We see that in general fourth-order statistics is needed. In the case when noise is Gaussian, we can reduce it to the two-order statistics [20]

$$\begin{aligned}&E\{n(t_1)\, n(t_2)\, n(t_3)\, n(t_4)\} \\ &\quad = E\{n(t_1)\, n(t_2)\}\, E\{n(t_3)\, n(t_4)\} + E\{n(t_1)\, n(t_3)\}\, E\{n(t_2)\, n(t_4)\} \\ &\quad\quad + E\{n(t_1)\, n(t_4)\}\, E\{n(t_2)\, n(t_3)\} \end{aligned} \tag{8}$$

so, we get

$$\begin{aligned}\sigma_{nn}^2 &= E\{|W_{nn}(\omega, t)|^2\} - |E\{W_{nn}(\omega, t)\}|^2 \\ &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[R_n^2\left(\frac{\xi - \zeta}{2}\right) + R_n^2\left(\frac{\xi + \zeta}{2}\right)\right] e^{-j\omega(\xi - \zeta)}\, d\xi\, d\zeta. \end{aligned} \tag{9}$$

Summing results

$$\sigma_{xx}^2 = \sigma_{nn}^2 + \sigma_{fn}^2. \tag{10}$$

When the noise is white, the autocorrelation function is $R_n(\tau) = \sigma_n^2\delta(\tau)$. The variance σ_{nn}^2 tends to infinite, and so does the variance $\sigma_{xx}^2 \to \infty$. The Wigner distribution of signals contaminated by this kind of noise cannot be used to estimate $W_{ff}(\omega, t)$.

Let us now consider the influence of truncating window commonly used in signal processing

$$\begin{aligned}x(t) &= w(t)[f(t) + n(t)] = w(t)f(t) + w(t)n(t) \\ &= f_w(t) + n_w(t). \end{aligned} \tag{11}$$

After calculation similar to the previous ones, we get the mean of Wigner distribution

$$\begin{aligned}E\{W_{xx}(\omega, t)\} &= W_{f_wf_w}(\omega, t) + S_{n_w}(\omega, t) \\ &= W_{ww}(\omega, t) \mathop{*}_{\omega} W_{ff}(\omega, t) + S_n(\omega) \mathop{*}_{\omega} W_{ww}(\omega, t) \end{aligned} \tag{12}$$

with $\mathop{*}_{\omega}$ denoting frequency domain convolution. The estimator is biased.

The variances, in this case, are

$$\begin{aligned}\sigma_{f_wn_w}^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_w(\xi, \zeta, t, t)\left\{\left[f\left(t + \frac{\xi}{2}\right)f^*\left(t + \frac{\zeta}{2}\right)\right.\right. \\ &\quad \left. + f^*\left(t - \frac{\xi}{2}\right)f\left(t - \frac{\zeta}{2}\right)\right] R_n\left(\frac{\xi - \zeta}{2}\right) \\ &\quad + \left[f^*\left(t - \frac{\xi}{2}\right)f^*\left(t + \frac{\zeta}{2}\right) + f\left(t + \frac{\xi}{2}\right)f\left(t - \frac{\zeta}{2}\right)\right] \\ &\quad \left.\cdot R_n\left(\frac{\xi + \zeta}{2}\right)\right\} e^{-j\omega(\xi - \zeta)}\, d\xi\, d\zeta. \end{aligned}$$

The second part of variance is

$$\begin{aligned}\sigma_{n_wn_w}^2 &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_w(\xi, \zeta, t, t)\left[R_n^2\left(\frac{\xi - \zeta}{2}\right) + R_n^2\left(\frac{\xi + \zeta}{2}\right)\right] \\ &\quad \cdot e^{-j\omega(\xi - \zeta)}\, d\xi\, d\zeta \end{aligned} \tag{13}$$

where

$$\begin{aligned}R_w(\xi, \zeta, \lambda, \mu) &= w(\lambda - \xi/2)w(\lambda + \xi/2) \\ &\quad \cdot w(\mu - \zeta/2)w(\mu + \zeta/2) \end{aligned} \tag{14}$$

If the noise is white, the variance $\sigma_{n_wn_w}^2$, also, tends to infinite. The estimation is not possible after smoothing in frequency domain, what is the case for Fourier transform of noisy signals [20]. Because of that we have to make smoothing with respect to the t axes and define the estimator of Wigner distribution in the form of smoothed pseudo-Wigner distribution [13], [18]

$$\begin{aligned}W_{xx}^s(\omega, t) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(t - \lambda)x(\lambda + \tau/2) \\ &\quad \cdot x^*(\lambda - \tau/2)e^{-j\omega\tau}\, d\tau\, d\lambda. \end{aligned} \tag{15}$$

The bias is

$$\begin{aligned}b &= W_{ff}(t, \omega) - E\{W_{xx}^s(\omega, t)\} \\ &= W_{ff}(t, \omega) - \left[p(t) \mathop{*}_{t} W_{f_wf_w}(t, \omega) + p(t) \mathop{*}_{t} S_n(\omega) \mathop{*}_{\omega} W_{ww}(\omega, t)\right] \end{aligned} \tag{16}$$

The last mod does not depend on the signal. For white noise it can be easily removed, so it will not be considered in approximative analysis.

The variance components are

$$\sigma^2_{f_w n_w} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \Big\{ p(t-\lambda)p(t-\mu)R_w(\xi, \zeta, \lambda, \mu) \cdot \Big[f\Big(\lambda + \frac{\xi}{2}\Big) f^*\Big(\mu + \frac{\zeta}{2}\Big) R_n\Big(\lambda - \mu - \frac{\xi - \zeta}{2}\Big) + f^*\Big(\lambda - \frac{\xi}{2}\Big) f\Big(\mu - \frac{\zeta}{2}\Big) R_n\Big(\lambda - \mu + \frac{\xi - \zeta}{2}\Big) + f^*\Big(\lambda - \frac{\xi}{2}\Big) f^*\Big(\mu + \frac{\zeta}{2}\Big) R_n\Big(\lambda - \mu + \frac{\xi + \zeta}{2}\Big) + f\Big(\lambda + \frac{\xi}{2}\Big) f\Big(\mu - \frac{\zeta}{2}\Big) R_n\Big(\lambda - \mu - \frac{\xi + \zeta}{2}\Big)\Big]\Big\} \cdot e^{-j\omega(\xi - \zeta)}\, d\xi\, d\zeta\, d\lambda\, d\mu \tag{17}$$

and

$$\sigma^2_{n_w n_w} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(t-\lambda)p(t-\mu)R_w(\xi, \zeta, \lambda, \mu) \cdot \Big[R_n\Big(\lambda - \mu + \frac{\xi - \zeta}{2}\Big) R_n\Big(\lambda - \mu - \frac{\xi - \zeta}{2}\Big) + R_n\Big(\lambda - \mu + \frac{\xi + \zeta}{2}\Big) R_n\Big(\lambda - \mu - \frac{\xi + \zeta}{2}\Big)\Big] \cdot e^{-j\omega(\xi - \zeta)}\, d\xi\, d\zeta\, d\lambda\, d\mu. \tag{18}$$

If the noise is white, then the component $\sigma^2_{n_w n_w}$, having in mind $\delta(\lambda - \mu + (\zeta \pm \xi)/2)\delta(\lambda - \mu - (\zeta \pm \xi)/2) = 0.5\delta(\lambda - \mu)\delta((\zeta \pm \xi)/2)$, can be written as

$$\sigma^2_{n_w n_w} = \sigma_n^4 \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p^2(t - \lambda) R_w(\tau, \tau, \lambda, \lambda) \cdot (1 + e^{j2\omega\tau})\, d\lambda\, d\tau. \tag{19}$$

We see that the variance is finite, time and frequency dependent, and dependent on the windows $w(\tau)$ and $p(t)$ shape. The windows are to satisfy the general conditions.[1]

Further, smoothing can be done, by the two-dimensional window $p(t, \omega)$ [21]. This type of window gives smaller variance and greater bias.

A. Approximative Analysis

If the window $p(t)$ is narrow enough comparing to the varying of signal $f(t)$ and window $w(\tau)$, then the variance of Wigner distribution can be approximated by

$$\sigma^2_{f_w n_w} \cong 2\sigma_n^2 \int_{-\infty}^{\infty} R_w(\tau, \tau, t, t) \Big\{ \Big| f\Big(t - \frac{\tau}{2}\Big)\Big|^2 + \Big| f\Big(t + \frac{\tau}{2}\Big)\Big|^2 + \Big[f^2\Big(t - \frac{\tau}{2}\Big) + f^{*2}\Big(t + \frac{\tau}{2}\Big)\Big] e^{j2\omega\tau}\Big\}\, d\tau \tag{20}$$

$$\sigma^2_{n_w n_w} \cong \sigma_n^4 p(0) \int_{-\infty}^{\infty} R_w(\tau, \tau, t, t)(1 + e^{j2\omega\tau})\, d\tau. \tag{21}$$

Previous expressions can be majored by

$$\sigma^2_{f_w n_w} \le 8\sigma_n^2 M_f^2 M_w^{(4)} \qquad \sigma^2_{n_w n_w} \le 2\sigma_n^4 p(0) M_w^{(4)} \tag{22}$$

where M_f^2 is the square magnitude of the signal $f(t)$, T is the window $w(\tau)$ width, and

$$M_w^{(4)} = \int_{-2T}^{2T} w^2(t - \tau/2) w^2(t + \tau/2)\, d\tau \le 2(2T - t). \tag{23}$$

In the signal processing of long signals, window $w(\tau)$ is often taken without shift. Thus we obtain

$$M_w^{(4)} = E_{w^2} = \int_{-2T}^{2T} w^4(\tau/2)\, d\tau \le 4T. \tag{24}$$

In that case, we will take that $w^2(\tau/2)$ is the window that satisfies general conditions.[1] If $w^2(\tau/2)$ is Hamming window, then $E_{w^2} = 1.59T$.

A rough estimation of variance can be done by

$$\sigma^2_{xx} \le 8\sigma_n^2 T[\sigma_n^2 p(0) + 4M_f^2]. \tag{25}$$

The bias of Wigner estimator, for long signals, can be written in the form

$$b = W_{ff}(\omega, t) - \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p(\lambda) w^2(\tau/2) f(t - \lambda + \tau/2) \cdot f^*(t - \lambda - \tau/2) e^{-j\omega\tau}\, d\tau\, d\lambda. \tag{26}$$

We will assume that the window $p(\lambda)$, as well as the Fourier transform $F_w(\theta)$ of window $w_e(\tau) = w^2(\tau/2)$, are narrow. After expansion of $f(t - \lambda + \tau/2)f^*(t - \lambda - \tau/2)$ into a Taylor series about the point $\lambda = 0$, and the $W_{ff}(\omega - \theta, t)$ about $\theta = 0$, neglecting the mods higher than third order, and after some transformations we get

$$\begin{aligned} -2b &\cong M_2^w \frac{\partial^2}{\partial\omega^2}[W_{ff}(\omega, t)] + M_2^p \frac{\partial^2}{\partial t^2}\Big\{ W_{ff}(\omega, t) + M_2^w \frac{\partial^2}{\partial\omega^2}[W_{ff}(\omega, t)]\Big\} \\ &\cong M_2^w \frac{\partial^2}{\partial\omega^2}[W_{ff}(\omega, t)] + M_2^p \frac{\partial^2}{\partial t^2}[W_{ff}(\omega, t)] \end{aligned} \tag{27a}$$

where $M_2^w = \int \omega^2 F_w(\omega)\, d\omega = -w_e''(0)$ and $M_2^p = \int t^2 p(t)\, dt = -P''(0)$ are energy and amplitude moments of windows $w^2(\tau/2)$ and $p(t)$. If $w^2(\tau/2)$ is Hamming window $M_2^w = 0.46\pi^2/(4T^2)$, and if $p(t)$ is rectangular $M_2^p = 1/(12p^2(0))$.

When the window $w(\tau)$ is taken with shift, then

$$b = W_{ff}(\omega, \tau)[1 - w^2(t)] - \frac{1}{2} M_2^{ww} \frac{\partial^2}{\partial\omega^2}[W_{ff}(\omega, t)] - \frac{1}{2} M_2^p \frac{\partial^2}{\partial t^2}[w^2(t) W_{ff}(\omega, t)] \tag{27b}$$

where $M_2^{ww} = \int \omega^2 W_{ww}(\omega, t)\, d\omega = -[w(t + \tau/2)w(t - \tau/2)]''_\tau$ for $\tau = 0$.

The ideal case with respect to the Wigner distribution bias is when $w(\tau) = 1$ for all τ and $p(t) = \delta(t)$. In that case we would not have a bias due to smoothing (which really had not been done in this case). The error due to the noise would be such that estimation cannot be done (9). The variance is smaller by narrowing the window $w(\tau)$ and widening the window $p(t)$, i.e., by increasing the smoothing in frequency and time domain, (25). This increasing leads to the increasing of bias. The choice has to be made by compromise between the bias and variance increasing of Wigner distribution estimator. The expressions (25) and (27) prove our comments on the windows shape. We see that variance is direct and bias inversely proportional to the width T of window $w(\tau)$, and to the amplitude $p(0)$ of window $p(t)$.

[1] The general conditions that are to be satisfied by window functions $w(0) = 1$ or $\int W(\omega)\, d\omega = 2\pi$; $P(\omega)|_{\omega=0} = 1$ or $\int p(t)\, dt = 1$.

The mean-square estimation error is $e = \sigma_{xx}^2 + b^2$. The optimal width T of window $w^2(\tau/2)$ and the optimal amplitude $p(0)$ of window $p(t)$, can be obtained from the condition that e is minimum. From the complete analysis the optimal window shapes could be derived. The minimum energy moment window $w^2(\tau/2)$ and the minimum amplitude moment window $p(t)$ minimize bias.

B. Analytic Signal

In the previous analysis, the real noise has been considered. To avoid the need for oversampling of signal by factor 2 and cross terms between positive and negative frequencies, the Wigner distribution is very often calculated by not using the original signal, but its analytic version. In that case, the noise is also "analytic" $\boldsymbol{n}_a(t) = \boldsymbol{n}(t) + j\hat{\boldsymbol{n}}(t)$. By $\hat{\boldsymbol{n}}(t)$ is denoted the Hilbert transform of noise $\boldsymbol{n}(t)$. The modifications in that case are as follows: the mean and the autocorrelation of Hilbert transform $\hat{\boldsymbol{n}}(t)$ are the same as for original noise $\boldsymbol{n}(t)$. The autocorrelation of noise $\boldsymbol{n}_a(t)$ is $R_{n_a}(\tau) = 2R_n(\tau) + j2R_n(\tau)*h(\tau)$, where $h(\tau) = 1/(\pi\tau)$ is the Hilbert transformer impulse response. Equation (8) for Gaussian noise holds in this case, because its Hilbert transform is also Gaussian. The expected value in the integral in (7) has the form $E\{\boldsymbol{n}_a(t_1)\boldsymbol{n}_a^*(t_2)\boldsymbol{n}_a^*(t_3)\boldsymbol{n}_a(t_4)\} = R_{n_a}(\xi)R_{n_a}(-\zeta) + R_{n_a}^2((\xi - \zeta)/2)$, which gives, replacing $R_n(\tau)$ by $R_{n_a}(\tau)$, an equation the same as (9) and (13), but without mod $R_{n_a}^2((\xi + \zeta)/2)$.

For white noise, knowing that $\mathrm{FT}[R_{n_a}(\tau)] = S_{n_a}(\omega) = 4\sigma_n^2 U(\omega)$, we get

$$\sigma_{n_a n_a}^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_{n_a}^2\left(\frac{\xi - \zeta}{2}\right) e^{-j\omega(\xi - \zeta)}\, d\xi\, d\zeta$$

$$= \frac{32}{\pi}\sigma_n^4 \omega\, U(\omega) \int_{-\infty}^{\infty} d\xi$$

$$\sigma_{n_{aw} n_{aw}}^2 = \frac{16}{\pi^2}\sigma_n^4 \int_0^{\infty} \theta\, W_{ww}^2(t, \omega - \theta)\, d\theta \qquad (28)$$

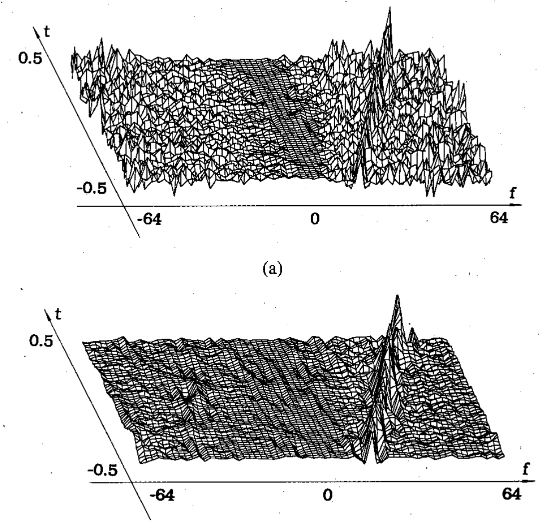
where $U(\omega)$ is a unite step function. We see that, when noise is analytic, variance can be finite with truncation window only. The variance is time and frequency dependent (for long signals, it is frequency dependent only).

III. Numerical Results

As examples we take linear and sinusoidal frequency modulated signals.

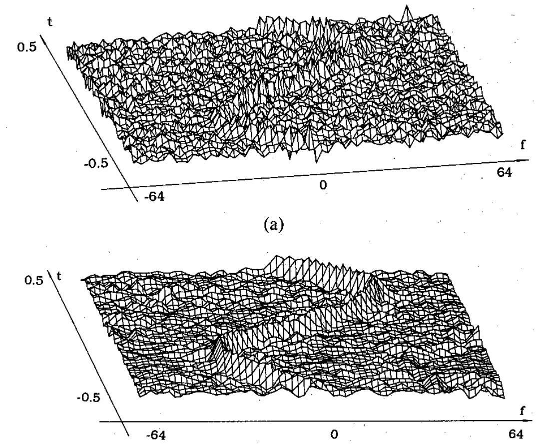
$$f(t) = \cos(at + bt^2/2), \quad \text{and} \quad f(t) = e^{-jd\cos(ct)} \qquad (29)$$

We consider the signals as being long. The influence of zero mean white Gaussian noise is taken into account. In the case of sinusoidal frequency modulated signal, the noise is taken to be real. For linear frequency modulated signal, the analytic version of signal and noise is taken. On the Fig. 1(a) and 2(a) the Wigner distribution of noisy signals with window $w(\tau)$ only is given. The Wigner distribution with both windows $w(\tau)$ and $p(t)$ is given on Fig. 1(b) and 2(b). The frequency dependence is remarkable on Fig. 1. The increasing of variance for negative frequencies, although not expected from (28), is due to the effects of discretization [22] used in computer simulation.



(b)

Fig. 1. Wigner distribution of linear frequency modulated analytic signal and Gaussian noise with Hamming window $w(\tau)$; $\sigma_n^2 = 0.5$; $a = 200$; $b = 200$; $|t| < 0.5$; $\Delta t = 1/256$; $N = 256$; $S/N = 3$ dB: (a) with $w(\tau)$ only; and (b) with $p(t) = 28.3 \cos^{0.2}(\pi t/0.04)$ for $|t| < 0.02$ (close to rectangular).



(b)

Fig. 2. Wigner distribution of sinusoidal frequency modulated signal and Gaussian noise with Hamming window $w(\tau)$; $\sigma_n^2 = 1$; $c = 2\pi$; $d = 32$; $|t| < 0.5$; $\Delta t = 1/256$; $N = 256$; $S/N = 0$ dB; (a) with $w(\tau)$ only; and (b) with $p(t, f) = 25.64 \cos^{0.2}(\pi t/0.02) \cos^{0.2}(\pi f/2.5)$ for $|t| < 0.01$ and $|f| < 1.25$.

IV. Conclusion

The analysis of noise influence to the Wigner distribution is done. It is shown that the Wigner distribution, calculated by standard definition, cannot be used to estimate Wigner distribution of signal without noise. In general, it is not sufficient to have commonly used smoothing in the frequency domain. The additional smoothing in time domain is necessary. If signal and noise are analytic then truncation can be sufficient to make the variance finite. The very simple expressions for variance and bias estimation are derived. The general characteristics of windows are pointed out. The results are demonstrated on the numerical examples.

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The Vector Split-Radix Algorithm for 2-D DHT

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Abstract—**In this correspondence, a decimation-in-frequency vector split-radix algorithm is proposed to decompose an $N * N$ 2-D DHT into one $(N/2) * (N/2)$ DHT and twelve $(N/4) * (N/4)$ DHT's. The proposed algorithm possesses the in-place property and needs no matrix transpose. Also, its computational structure is very regular and is simpler than those of all existing nonseparable 2-D DHT's.**

I. Introduction

The one-dimensional (1-D) split-radix approach [1], [2] applies a radix-2 decomposition to the even indexed samples and a radix-4 decomposition to the odd indexed samples. Recently, Duhamel [3] showed that split-radix fast Fourier transform (FFT) is the best possible tradeoff between the arithmetic complexity and the structural regularity for length-2^n FFT's. In two-dimensional (2-D) problems, the vector radix method is available [4]. In this method, a 2-D discrete Fourier transform (DFT) is divided into successively smaller 2-D DFT's until, ultimately, only trivial 2-D DFT's needed to be evaluated. It has been proved that the vector radix method needs 25% fewer complex multiplications than the conventional row-column decomposition. Recently, the split-radix FFT algorithm has been extended to two dimensions using decimation in frequency (DIF) [5] and decimation in time [6]. Further, in [7], the 2-D vector split-radix DIF FFT algorithm was derived using a structural approach.

Although symmetries of the DFT of a real-valued sequence can be exploited to reduce both the storage and the computational costs, a transform that directly maps a real-valued sequence to a real-valued spectrum while preserving most of the useful properties of the DFT is sometimes preferred. One such transform is the discrete Hartley transform [8]. Sorensen *et al.* [9] have developed a complete set of fast Hartley transform (FHT) algorithms, including DIF radix-2, radix-4, split-radix, prime factor, and Winograd–Hartley transform algorithms. Although both [9] and [10] indicated the fact that it is hopeless to find FHT algorithms requiring fewer arithmetic operations than the corresponding FFT algorithms, the self-inverse property of DHT makes it interesting in some applications, such as spectral analysis [11]. Thus, many works have been devoted to the development of better and faster algorithms for 1-D DHT [8]–[14]. The major concern for the extension of 1-D DHT to 2-D is the separability of 2-D DHT [15]–[17]. Huang *et al.* [18] have derived a vector split-radix algorithm for 2-D nonseparable DHT in which an $N * N$ 2-D DHT is decomposed into three $(N/2) * (N/2)$ DHT's and four $(N/4) * (N/4)$ DHT's, as the one given in [5] developed for computing 2-D DFT.

In this correspondence, a DIF vector split-radix algorithm is proposed to decompose an $N * N$ 2-D DHT into one $(N/2) * (N/2)$ DHT and twelve $(N/4) * (N/4)$ DHT's, as the one was done for 2-D DFT given in [7]. The proposed algorithm possesses the

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